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Third-order safe consensus of heterogeneous vehicular platoons with MPF network topology: constant time headway strategy

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Abstract

In this paper the problem of centralized control of a platoon of non-identical vehicles under constant time headway strategy (CTHS) is investigated for multi predecessors following (MPF) topology. A centralized neighbor based linear control law using relative position and velocity is considered for each vehicle. Due to communication and parasitic delays and time-varying network topology, the closed-loop dynamics of platoon is in the form of a multiple delayed switched linear system. New approaches are developed to perform the internal stability analysis of one-dimensional heterogeneous vehicular platoons. Afterwards, sufficient conditions assuring the string stability of a platoon under MPF topology are obtained by presenting a new theorem. In continuance of the paper, some conditions on control parameters guaranteeing safety of the platoon in an emergency braking maneuver are presented through a new theorem. Several simulation results are provided to show the effectiveness of the proposed methods.

Keywords

Platoon of vehicles, time-varying delay, parasitic delay, time-varying network, internal stability, string stability, collision avoidance

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Introduction

In recent decades, the problem of traffic congestion has received great attention as a serious social, environmental, and economic problem.[1–5] As a result, the intelligent transportation system (ITS) idea, as a possible solution for this problem, has been highly regarded.[4,5] The main objective of an ITS is maintaining small inter-vehicle spacing in vehicular platoons.[6,7] There are several benefits in implementing an ITS, such as enhancement of safety, increasing highways capacity and fuel efficiency, and decreasing air pollution.[2,8,9]

The coordinated motion of a group of vehicles moving with optimal spacing and common velocity is called vehicular platooning.[10] Vehicular platooning is a useful tool to implement the idea of ITS.[1,11–13] Vehicular platooning has received a lot of attention in recent decades. As a result, several methods are provided for control design and stability analysis of one-dimensional (1-D) vehicular platoons.[14–20]

In vehicular platooning, in addition to usual stability analysis, the string stability analysis is also considered. In a platoon, vehicles are dynamically connected by feedback control laws. Therefore, the spacing error created by each vehicle, affects others which may propagate upstream the platoon. This phenomenon is called string instability.[21,22] The inter-vehicle spacing between consecutive vehicles are adjusted by two different strategies: constant spacing strategy (CSS);[21,23,24] and constant time headway strategy (CTHS).[19–21] In CSS the inter-vehicle spacing is constant, but in CTHS it varies with velocity.

In recent decades, a great deal of research works has been carried out on vehicular platooning. Linear controllers without considering parameter uncertainties have been applied.[9,11,13,17] In other studies,[21,25] adaptive controllers are designed to estimate the unknown parameters such as rolling resistance, air drag force,

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and vehicle mass. Stability analysis using a partial differential equation approximation of a third-order dynamical model has been employed.[15,19] Homogenous platoons have been investigated,[19,22,24] while heterogeneous platoons have also been studied.[15,17] A model predictive control is designed to guarantee the internal and string stability of vehicular platoons.[26] A leader-predecessor following topology with CSS was studied.[12,13] Bidirectional schemes for CSS and CTHS have also been studied.[15,19,25] A centralized controller based on a leader-predecessor following scheme with experimental validations is presented.[13] A comparison between the performances of different information flows on stability margin is presented.[14] The communication delay is considered in stability analysis of platoons.[12,13,15,17,27]

The third-order consensus of heterogeneous vehicular platoons has not been studied in previous works. Therefore, the effect of time-varying interaction topology on internal and string stability of vehicular platoons has not been investigated for platoons with a third-order model of upper level dynamics. Moreover, the collision avoidance and string stability problems of vehicular platoons with MPF topology in presence of communication and parasitic delays have not been studied so far.

In this paper, a safe control methodology of 1-D heterogeneous vehicular platoons is investigated. A third-order dynamical model is used to describe the longitudinal motion of each vehicle. The network topology is assumed to be variable by time. Both communication and parasitic delays are involved in system modeling and controller design. By applying Lyapunov–Razumikhin and Lyapunov–Krasovskii theorems, new approaches for constructing common Lyapunov functions (CLFs) for the resultant switched linear multiple delays system are introduced for CTHS.

In the existing literature, the string stability of vehicular platoons with multi predecessors following (MPF) topology in the presence of communication and parasitic delays has not been studied. In the study by Jia and Ngoduy,[23] only internal stability of a platoon with MPF topology under communication delay was investigated. The controller presented by Xiao and Gao cannot satisfy the string stability of MPF topology in the presence of a delay.[11] Therefore, another important objective of this paper is to solve the problem of string stability of vehicular platoons with MPF topology under communication and parasitic delays with switched network topology. In brief, the main innovations of this paper are as follows:

(1) Presenting a modified Razumikhin-based approach for stability analysis of third-order switched linear time delay systems. In previous and similar studies,[19,23] the Razumikhin theorem for stability analysis of switched networks is incorrectly applied, which leads to incorrect results and a fundamental contradiction.
(2) String stability analysis of third-order heterogeneous vehicular platoons with MPF topology by considering communication and parasitic delays.
(3) Presenting a Krasovskii-based method to find a CLF for stability analysis of vehicular platoons under time-varying delays by using the concepts of switching systems.[28]
(4) Presenting a robust safe consensus protocol guaranteeing collision avoidance against communication and parasitic delays.

The rest of paper is organized as follows. In the following section, mathematical preliminaries are introduced briefly. The third section discusses the third-order longitudinal vehicle model briefly. Also, the controller design and internal stability analysis are presented in this section. String stability of heterogeneous vehicular platoons in the presence of communication and parasitic delays and switching topology is then discussed. The collision avoidance problem of a platoon is studied analytically. Simulation studies are provided to show the effectiveness of the proposed approaches. Finally, this paper is then concluded in the final section.

Graph theory and mathematical lemmas

Let $G = (V, E, \mathbf{A})$ be a graph of order N in which $V = \{1, 2, \ldots, N\}$ represents a node set, $E \subseteq N \times N$ is the set of edges, and $\mathbf{A}$ is the adjacency matrix with nonnegative elements. An edge (i,j) denotes that the node j has access to the information of the node i. Set of neighbors of node i is shown by $N_i = \{j \in V : (j, i) \in \varepsilon, j \neq i\}$. In the leader-follower scheme, for the follower agents 1 to N, there exists a leader labeled by 0. Information is exchanged between the leader and the follower agents which belong to the neighbors of the leader. Then, the graph $\bar{G} = (\bar{V}, \bar{E}, \mathbf{A})$ with node set $\bar{V} = V \cup \{0\}$ and edge set $\bar{E} = \bar{V} \times \bar{V}$ represents the communication topology between the leader and the followers. A diagonal matrix $\mathbf{B} \in \mathbb{R}^{N \times N}$ is defined as a leader adjacency matrix of $\bar{G}$ with diagonal elements $b_i = a_{i0}$. If lead vehicle is a neighbor of vehicle i, $a_{i0} > 0$ and $a_{i0} = 0$ otherwise. Node 0 is globally reachable in $\bar{G}$ if there is a path form every node $i \in V$ to it. For graph G the Laplacian matrix $\mathbf{L} = [l_{ij}] \in \Re^{N \times N}$ is defined with $l_{ii} = \sum_{j=1, \neq i}^{N} a_{ij}$ and $l_{ij} = -a_{ij}$, $i \neq j$. Also, for graph $\bar{G}$ the important matrix $\mathbf{H} = \mathbf{L} + \mathbf{B}$ is defined.

Lemma 1.[29] $\mathbf{H} \succ \mathbf{0}$ if and only if the lead vehicle is globally reachable in $\bar{G}$.

Lemma 2.[30] The symmetric matrix $\mathbf{M} = \begin{pmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21}^T & \mathbf{M}_{22} \end{pmatrix}$ is positive definite if and only if: $\mathbf{M}_{11} \succ \mathbf{0}$ and $\mathbf{M}_{22} - \mathbf{M}_{12}^T \mathbf{M}_{11}^{-1} \mathbf{M}_{12} \succ \mathbf{0}$.

Lemma 3.[31] For any vectors $\boldsymbol{\delta}_1$, $\boldsymbol{\delta}_2$ and any positive definite matrix $\boldsymbol{\Psi}$, the inequality $2\boldsymbol{\delta}_1^T \boldsymbol{\delta}_2 \leqslant \boldsymbol{\delta}_1^T \boldsymbol{\Psi} \boldsymbol{\delta}_1 + \boldsymbol{\delta}_2^T \boldsymbol{\Psi}^{-1} \boldsymbol{\delta}_2$ holds.

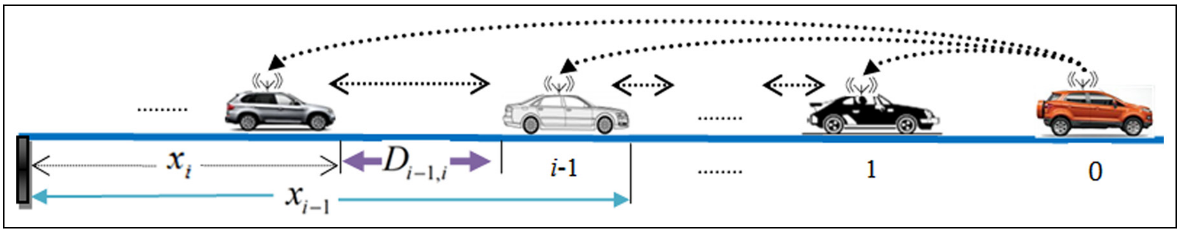

Figure 1. Fully connected network of a 1-D heterogeneous platoon of vehicles.

Lemma 4.[30] Suppose that $\lambda_i, i = 1, \ldots, n$ are the eigenvalues of $\Xi \in \mathbb{R}^{n\times n}$ and κ_i, $i = 1, 2, \ldots, m$ are the eigenvalues of $\Upsilon \in \mathbb{R}^{m\times m}$. So that, the eigenvalues of $\Xi \otimes \Upsilon$ are $\lambda_1\kappa_1, \ldots, \lambda_1\kappa_m, \lambda_2\kappa_1, \ldots, \lambda_2\kappa_m, \ldots, \lambda_n\kappa_1, \ldots, \lambda_n\kappa_m$, where $\otimes$ is the Kronecker product.

String stability:[21] A platoon of vehicles is string stable if the spacing errors between consecutive vehicles do not amplify along the string. String stability of N vehicles can be met with the following requirement:

$$\|e_1\|_\infty \geqslant \|e_2\|_\infty \geqslant \ldots \geqslant \|e_N\|_\infty \tag{1}$$

where e_i is the spacing error of ith vehicle.

Problem description

The longitudinal motion of vehicular platoons consisting of a lead vehicle and N non-identical followers is investigated. It is assumed that each vehicle is equipped with GPS and a wireless as shown in Figure 1. Therefore, each vehicle can measure its own absolute position and velocity and has access to its neighbors' absolute position.

The longitudinal dynamics of the ith vehicle is modeled by the following nonlinear equation:[14]

$$\dot{a}_i = f_i(v_i, a_i) + g_i(v_i)c_i \tag{2}$$

where x_i, v_i, and a_i are position, velocity, and acceleration of the ith vehicle, respectively, and c_i is the input of engine. Also, $f_i(v_i, a_i)$ and $g_i(v_i)$ are as follows:

$$f_i(v_i, a_i) = -\frac{1}{T_i}\left(a_i + \frac{\sigma A_i c_{di}}{2m_i}v_i^2 + \frac{R_i}{m_i}\right) - \frac{\sigma A_i c_{di} v_i a_i}{m_i}, \quad g_i(v_i) = \frac{1}{T_i m_i} \tag{3}$$

where σ. is density of air, T_i, A_i, c_{di}, R_i, and m_i are engine time constant, cross-sectional area, air drag coefficient, rolling resistance force, and mass of ith vehicle, respectively. By adopting the following control law:

$$c_i = u_i m_i + 0.5\sigma A_i c_{di} v_i^2 + R_i + T_i \sigma A_i c_{di} v_i a_i \tag{4}$$

where u_i is the additional control input, the following third-order linear differential equation is obtained:

$$T_i \dot{a}_i + a_i = u_i \tag{5}$$

In general, the control architecture of a vehicle is composed of two levels: the lower level control which compensates the nonlinear vehicle dynamics and the upper level control which designs the desired acceleration of vehicle. In this paper, only upper level control is designed and it is assumed that the lower control has already been designed. The model of equation (5) has been extensively used in upper level control design and stability analysis of the longitudinal vehicle's motion.[6,13,19,24,27]

As will be shown in stability analyses, the lead vehicle should be globally reachable in the platoon. Therefore, each vehicle has access to the lead vehicle's position and velocity (directly from lead vehicle or through other neighbors). By considering communication delay, the following control law is considered for the ith vehicle:

$$\begin{aligned} u_i = &-D\left[v_i(t) - b_i^\sigma v_0(t - \tau_{i0}(t)) - (1 - b_i^\sigma)v_0(t - \tau_{i0}(t))\right] - \\ &-K\left[x_i(t) - x_0(t - \tau_{i0}(t)) - \tau_{i0} b_i^\sigma v_0(t - \tau_{i0}(t)) - \tau_{i0}(1 - b_i^\sigma)v_0(t - \tau_{i0}(t)) - d_{0i}\right] - \\ &-K\sum_{j=1}^{N} a_{ij}^\sigma\left[x_i(t) - x_j\left(t - \tau_{ij}(t)\right) - \tau_{ij}v_0\left(t - \tau_{ij}(t)\right) - d_{ij}\right] \end{aligned} \tag{6}$$

In vehicular platooning, it is assumed that the lead vehicle has a constant velocity during motion. So that, the above control law can be expressed as follows:

$$\begin{aligned} u_i = &-D(v_i - v_0) - K[x_i(t) - x_0(t - \tau_{i0}(t)) - \tau_{i0}v_0 - d_{0i}] \\ &-K\sum_{j=1}^{N} a_{ij}^\sigma\left[x_i(t) - x_j\left(t - \tau_{ij}(t)\right) - \tau_{ij}v_0 - d_{ij}\right] \end{aligned} \tag{7}$$

where $\tau_{ij}(t)$ is the time-varying communication delay between vehicles i and j. K and D are parameters of controller. $\sigma(t) : [0, \infty) \to k \in \{1, 2, \ldots, n_s\}$ is the switching signal and n_s is the number of subsystems. The desired position of the ith vehicle is defined as $x_i^d = x_0 - d_{0i}$. It is defined that $d_{ij} = \sum_{k=j}^{i-1} [h_k(v_0 - \bar{v}) + L_k] + (i-j)D_{\min}$, where h_k and L_k are constant time headway and length of the ith vehicle, $D_{\min}$ is the minimum allowable inter-vehicle spacing, and $\bar{v}$ is a constant value. Since the velocity of lead vehicle is constant, the tracking error and its time derivative are as follows:

$$e_i = x_i - x_i^d \Rightarrow \dot{e}_i = \dot{x}_i - v_0 \Rightarrow \ddot{e}_i = a_i \Rightarrow \dddot{e}_i = \dot{a}_i \quad (8)$$

The control input of equation (7) can be written in terms of tracking error as:

$$u_i = -D\dot{e}_i - Ke_i - K\sum_{j=1}^{N} a_{ij}^{\sigma}\left[e_i(t) - e_j\left(t - \tau_{ij}(t)\right)\right] \quad (9)$$

Due to parasitic delay, the term of control law $u_i(t)$ is replaced by $u_i(t - \Delta_i)$. By considering parasitic delay and inserting equation (9) in equation (5), the closed-loop dynamics of the ith vehicle is obtained as follows:

$$\begin{aligned} T_i\dddot{e}_i + \ddot{e}_i = & -D\dot{e}_i(t-\Delta_i) - Ke_i(t-\Delta_i) \\ & - K\sum_{j=1}^{N} a_{ij}^{\sigma}\left[e_i(t-\Delta_i) - e_j(t - \bar{\tau}_{ij}(t))\right] \end{aligned} \quad (10)$$

where $\bar{\tau}_{ij}(t) = \tau_{ij}(t) + \Delta_i$. Equation (9) plays the role of upper level control and equation (4) is the lower level control of each vehicle. Figure 2 depicts the relation between the upper level and lower level controls.

By defining the error vector as $\mathbf{e} = [e_1, \ldots, e_N, \dot{e}_1, \ldots, \dot{e}_N, \ddot{e}_1, \ldots, \ddot{e}_N]^T$, the closed-loop dynamics of platoon is represented in the following form

$$\dot{\mathbf{e}} = \mathbf{A}_\sigma \mathbf{e} + \sum_{i=1}^{N} \mathbf{B}_{i,\sigma}(t - \Delta_i) + \sum_{r=1}^{\bar{m}} \mathbf{C}_{r,\sigma}\mathbf{e}(t - \bar{\tau}_r(t)), \quad \bar{m} \leqslant N(N-1) \quad (11)$$

where $\bar{\tau}_r(t) = \{\bar{\tau}_{ij}(t) : i,j = 1, \ldots, N, i \neq j\}$ and

$$\mathbf{A}_\sigma = \begin{pmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{I}} \end{pmatrix}, \quad \mathbf{B}_{i,\sigma} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -K(\bar{\mathbf{I}} + \bar{\mathbf{D}}_{i,\sigma}) & -D\bar{\mathbf{I}}_i & \mathbf{0} \end{pmatrix},$$

$$\mathbf{C}_{r,\sigma} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -K\bar{\mathbf{C}}_{r,\sigma} & -D\bar{\mathbf{I}}_i & \mathbf{0} \end{pmatrix}$$

$$\bar{\mathbf{I}} = diag\{1/T_1, 1/T_2, \ldots, 1/T_N\},$$
$$\bar{\mathbf{D}}_{i,\sigma} = diag\{0, \ldots, 0, d_i^{\sigma}/T_i, 0, \ldots, 0\},$$
$$\bar{\mathbf{I}}_i = diag\{0, \ldots, 0, 1/T_i, 0 \ldots, 0\},$$
$$\left[\bar{\mathbf{C}}_{r,\sigma}\right]_{jk} = \begin{cases} a_{jk}^{\sigma}/T_j, & j \neq k, \bar{\tau}_r(t) = \bar{\tau}_{jk}(t) \\ 0, & otherwise \end{cases}$$

Theorem 1. If the following conditions are satisfied, the 1-D heterogeneous platoon of vehicles is internal stable under arbitrary switching.

1. The lead vehicle is globally reachable in all subsystems.
2. The following conditions hold:

$$\frac{D}{T_{\max}^2} - K\bar{\lambda}_R > 0, \ \underline{\lambda}_R\left[K\,\underline{\lambda}_R - \frac{D}{T_{\max}^2}\right]^2 - \frac{1}{T_{\min}^3}\,\underline{\lambda}_I^2 > 0 \quad (12)$$

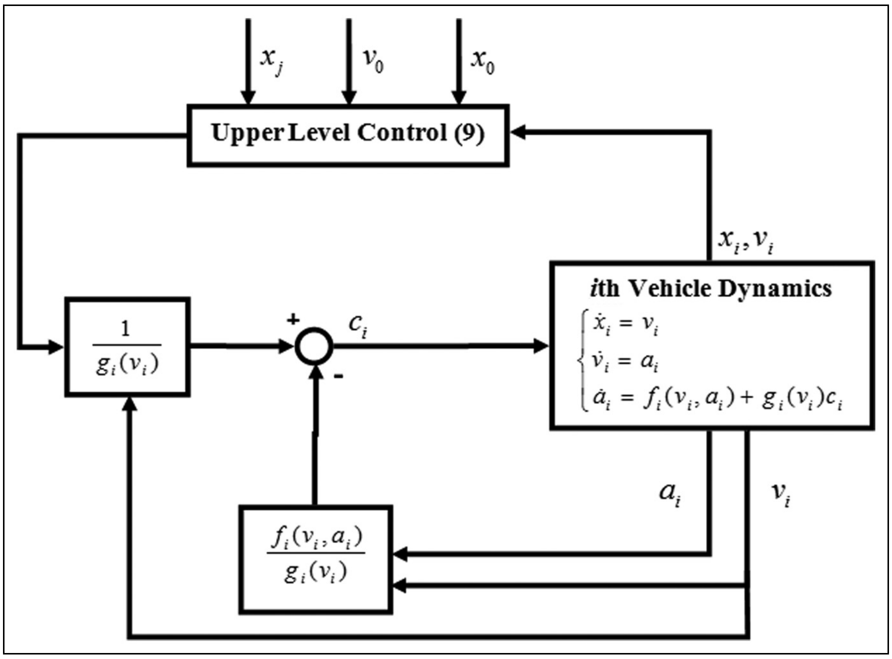

Figure 2. The block diagram of a vehicle consisting of the upper level (u_i) and the lower level (c_i) controls.

where $\bar{\lambda}_R = \max_{i=1,\dots,N}\left\{\max_{i=1,\dots,n_s}\mathrm{Re}(\lambda_{i,k})\right\}$, $\underline{\lambda}_R = \min_{i=1,\dots,N}\left\{\min_{i=1,\dots,n_s}\mathrm{Re}(\lambda_{i,k})\right\}$, $\bar{\lambda}_R = \bar{\lambda}_R = \max_{i=1,\dots,N}\left\{\max_{i=1,\dots,n_s}\mathrm{Im}(\lambda_{i,k})\right\}$, $\lambda_{i,k}$ is the ith eigenvalue of $\bar{\mathbf{H}}_k$ and $\bar{\mathbf{H}}_k = \bar{\mathbf{I}} + \sum_{i=1}^{N}\bar{\mathbf{D}}_{i,k} - \sum_{r=1}^{\bar{m}}\bar{\mathbf{C}}_{r,k}$. It can be easily shown that for a homogeneous platoon $\bar{\mathbf{H}}_k = \mathbf{H}_k = \mathbf{L}_k + \mathbf{I}$.

3. There exists a symmetric positive definite matrix $\mathbf{P} \in \mathbb{R}^{N\times N}$ such that the following inequality holds:

$$\mathbf{P}\mathbf{F}_k + \mathbf{F}_k^T\mathbf{P} \prec -\mathbf{Q} \prec \mathbf{0}, \quad \mathbf{F}_k = \mathbf{A}_k + \sum_{i=1}^{N}\mathbf{B}_{i,k} + \sum_{r=1}^{\bar{m}}\mathbf{C}_{r,k},\ k \in \{1, 2, \dots, n_s\} \tag{13}$$

Proof. A switching system under arbitrary switching is stable if all subsystems are stable and share a CLF. So that, the necessary condition for stability of system of equation (11) is that $\mathbf{F}_k \prec \mathbf{0}$. The characteristic equation of $\mathbf{F}_k$ can be written in the following form:

$$\begin{aligned}\det(s\mathbf{I} - \mathbf{F}_k) &= \det\begin{pmatrix} s\mathbf{I}_N & -\mathbf{I}_N & \mathbf{0} \\ \mathbf{0} & s\mathbf{I}_N & -\mathbf{I}_N \\ K\bar{\mathbf{H}}_k & D\bar{\mathbf{I}}_N & s\mathbf{I}_N + \bar{\mathbf{I}}_N \end{pmatrix} \\ &= \det\left(s^3\mathbf{I}_N + s^2\bar{\mathbf{I}}_N + sD\bar{\mathbf{I}}_N + K\bar{\mathbf{H}}_k\right) = \\ &= \prod_{i=1}^{N}\left(s^3 + \frac{1}{T_i}s^2 + \frac{D}{T_i}s + K\lambda_{i,k}\right) \\ &= \prod_{i=1}^{N}\zeta_{i,k}(s, \lambda_{i,k})\end{aligned} \tag{14}$$

$\mathrm{Re}(\lambda_{i,k}) > 0$ if the lead vehicle is globally reachable for all k. The Bilharz matrix associated to $\zeta_{i,k}(s, \lambda_{i,k})$ is in the following form:[32]

$$\mathbf{B}_M = \begin{pmatrix} 1 & 0 & -D/T_i & -K\,\mathrm{Im}(\lambda_{i,k}) & 0 & 0 \\ 0 & 1/T_i & 0 & -K\,\mathrm{Re}(\lambda_{i,k}) & 0 & 0 \\ 0 & 1 & 0 & -D/T_i & -K\,\mathrm{Im}(\lambda_{i,k}) & 0 \\ 0 & 0 & 1/T_i & 0 & -K\,\mathrm{Re}(\lambda_{i,k}) & 0 \\ 0 & 0 & 1 & 0 & -D/T_i & -K\,\mathrm{Im}(\lambda_{i,k}) \\ 0 & 0 & 0 & 1/T_i & 0 & -K\,\mathrm{Re}(\lambda_{i,k}) \end{pmatrix}$$

The even-order minors of $\mathbf{B}_M$ are:

$$\begin{aligned} E_1 &= \det\begin{pmatrix} 1 & 0 \\ 0 & 1/T_i \end{pmatrix}, \\ E_2 &= \det\begin{pmatrix} 1 & 0 & -D/T_i & -K\,\mathrm{Im}(\lambda_{i,k}) \\ 0 & 1/T_i & 0 & -K\,\mathrm{Re}(\lambda_{i,k}) \\ 0 & 1 & 0 & -D/T_i \\ 0 & 0 & 1/T_i & 0 \end{pmatrix}, \\ E_3 &= \det(\mathbf{B}_M) \end{aligned} \tag{15}$$

By doing some algebraic calculations, we have:

$$\begin{aligned} &E_1 = \frac{1}{T_i},\ E_2 = \frac{D}{T_i^3} - \frac{K}{T_i}\mathrm{Re}(\lambda_{i,k}), \\ &E_3 = K\,\mathrm{Re}(\lambda_{i,k})\left[K\,\mathrm{Re}(\lambda_{i,k}) - \frac{D}{T_i^2}\right]^2 - \frac{K}{T_i^3}(\mathrm{Im}(\lambda_{i,k}))^2 \end{aligned}$$

All minor values of $E_i, i = 1, 2, 3$ are positive if all conditions in equation (12) are satisfied. In continuance of the proof, consider the following CLF and its time derivative along equation (11):

$$V = \mathbf{e}^T\mathbf{P}\mathbf{e} \Rightarrow \dot{V} = 2\mathbf{e}^T\mathbf{P}\left\{\mathbf{A}_{\sigma(t)}\mathbf{e} + \sum_{i=1}^{N}\mathbf{B}_{i,\sigma(t)}\mathbf{e}(t - \Delta_i) + \sum_{r=1}^{\bar{m}}\mathbf{C}_{r,\sigma(t)}\mathbf{e}(t - \bar{\tau}_r(t))\right\} \tag{16}$$

By using the Newton–Leibnitz formula:[33,34]

$$\begin{aligned} \mathbf{e}(t - \Delta_i) &= \mathbf{e}(t) - \int_{t-\Delta_i}^{t}\dot{\mathbf{e}}(s)ds, \quad \mathbf{e}(t - \bar{\tau}_r) \\ &= \mathbf{e}(t) - \int_{t-\bar{\tau}_r}^{t}\dot{\mathbf{e}}(s)ds \end{aligned} \tag{17}$$

$\dot{V}$ can be written in the following form:

$$\begin{aligned} \dot{V} = 2\mathbf{e}^T\mathbf{P}\Bigg\{&\mathbf{A}_{\sigma(t)}\mathbf{e} + \sum_{i=1}^{N}\mathbf{B}_{i,\sigma(t)}\left[\mathbf{e}(t) - \int_{t-\Delta_i}^{t}\dot{\mathbf{e}}(s)ds\right] \\ &+ \sum_{r=1}^{\bar{m}}\mathbf{C}_{r,\sigma(t)}\left[\mathbf{e}(t) - \int_{t-\bar{\tau}_r}^{t}\dot{\mathbf{e}}(s)ds\right]\Bigg\} \end{aligned} \tag{18}$$

Equation (18) can be simplified as:

$$\dot{V} = \mathbf{e}^T\left\{\mathbf{P}\left(\mathbf{A}_{\sigma(t)} + \sum_{i=1}^{N}\mathbf{B}_{i,\sigma(t)} + \sum_{r=1}^{\bar{m}}\mathbf{C}_{r,\sigma(t)}\right) + \left(\mathbf{A}_{\sigma(t)} + \sum_{i=1}^{N}\mathbf{B}_{i,\sigma(t)} + \sum_{r=1}^{\bar{m}}\mathbf{C}_{r,\sigma(t)}\right)^T\mathbf{P}\right\}\mathbf{e} + \\ + \sum_{i=1}^{N}\int_{t-\Delta_i}^{t} -2\mathbf{e}^T\mathbf{P}\mathbf{B}_{i,\sigma(t)}\left\{\mathbf{A}_{\sigma(s)}\mathbf{e}(s) + \sum_{j=1}^{N}\mathbf{B}_{j,\sigma(s)}\mathbf{e}(s-\Delta_j) + \sum_{j=1}^{\bar{m}}\mathbf{C}_{j,\sigma(s)}\mathbf{e}(s-\bar{\tau}_j(t))\right\}ds + \\ + \sum_{i=1}^{\bar{m}}\int_{t-\bar{\tau}_i}^{t} -2\mathbf{e}^T\mathbf{P}\mathbf{C}_{i,\sigma(t)}\left\{\mathbf{A}_{\sigma(s)}\mathbf{e}(s) + \sum_{j=1}^{N}\mathbf{B}_{j,\sigma(s)}\mathbf{e}(s-\Delta_j) + \sum_{j=1}^{\bar{m}}\mathbf{C}_{j,\sigma(s)}\mathbf{e}(s-\bar{\tau}_j(t))\right\}ds \tag{19}$$

Since $\mathbf{B}_{i,\sigma(t)}\mathbf{B}_{j,\sigma(s)} = \mathbf{B}_{i,\sigma(t)}\mathbf{C}_{j,\sigma(s)} = \mathbf{C}_{i,\sigma(t)}\mathbf{B}_{j,\sigma(s)} = \mathbf{C}_{i,\sigma(t)}\mathbf{C}_{j,\sigma(s)} = \mathbf{0}$, $\dot{V}$ can be written in the following form:

$$\dot{V} = \mathbf{e}^T\left\{\mathbf{P}\left(\mathbf{A}_{\sigma(t)} + \sum_{i=1}^{N}\mathbf{B}_{i,\sigma(t)} + \sum_{r=1}^{\bar{m}}\mathbf{C}_{r,\sigma(t)}\right) + \left(\mathbf{A}_{\sigma(t)} + \sum_{i=1}^{N}\mathbf{B}_{i,\sigma(t)} + \sum_{r=1}^{\bar{m}}\mathbf{C}_{r,\sigma(t)}\right)^T\mathbf{P}\right\}\mathbf{e} + \\ + \sum_{i=1}^{N}\int_{t-\Delta_i}^{t} -2\mathbf{e}^T\mathbf{P}\mathbf{B}_{i,\sigma(t)}\mathbf{A}_{\sigma(s)}\mathbf{e}(s)ds + \sum_{i=1}^{\bar{m}}\int_{t-\bar{\tau}_i}^{t} -2\mathbf{e}^T\mathbf{P}\mathbf{C}_{i,\sigma(t)}\mathbf{A}_{\sigma(s)}\mathbf{e}(s)ds \tag{20}$$

By exploiting lemma 3 in the following form:

$$\begin{aligned} &-2\mathbf{e}^T(t)\mathbf{P}\mathbf{B}_{i,\sigma(t)}\mathbf{A}_{\sigma(s)}\mathbf{e}(s): \\ &\boldsymbol{\delta}_1^T = -\mathbf{e}^T(t)\mathbf{P}\mathbf{B}_{i,\sigma(t)}\mathbf{A}_{\sigma(s)};\ \boldsymbol{\delta}_2 = \mathbf{e}(s); \quad \boldsymbol{\Psi} = \mathbf{P}^{-1} \\ &-2\mathbf{e}^T(t)\mathbf{P}\mathbf{C}_{i,\sigma(t)}\mathbf{A}_{\sigma(s)}\mathbf{e}(s): \\ &\boldsymbol{\delta}_1^T = -\mathbf{e}^T(t)\mathbf{P}\mathbf{C}_{i,\sigma(t)}\mathbf{A}_{\sigma(s)};\ \boldsymbol{\delta}_2 = \mathbf{e}(s); \quad \boldsymbol{\Psi} = \mathbf{P}^{-1} \end{aligned} \tag{21}$$

equation (20) can be expressed as:

$$\dot{V} \leqslant \mathbf{e}^T\left\{\mathbf{P}\left(\mathbf{A}_{\sigma(t)} + \sum_{i=1}^{N}\mathbf{B}_{i,\sigma(t)} + \sum_{r=1}^{\bar{m}}\mathbf{C}_{r,\sigma(t)}\right) + \left(\mathbf{A}_{\sigma(t)} + \sum_{i=1}^{N}\mathbf{B}_{i,\sigma(t)} + \sum_{r=1}^{\bar{m}}\mathbf{C}_{r,\sigma(t)}\right)^T\mathbf{P}\right\}\mathbf{e} + \\ + \sum_{i=1}^{N}\int_{t-\Delta_i}^{t}\left(\mathbf{e}^T(s)\mathbf{P}\mathbf{e}(s) + \mathbf{e}^T(t)\mathbf{P}\mathbf{B}_{i,\sigma(t)}\mathbf{A}_{\sigma(s)}\mathbf{P}^{-1}\mathbf{A}_{\sigma(s)}^T\mathbf{B}_{i,\sigma(t)}^T\mathbf{P}\mathbf{e}(t)\right)ds + \\ + \sum_{i=1}^{\bar{m}}\int_{t-\bar{\tau}_i}^{t}\left(\mathbf{e}^T(s)\mathbf{P}\mathbf{e}(s) + \mathbf{e}^T(t)\mathbf{P}\mathbf{C}_{i,\sigma(t)}\mathbf{A}_{\sigma(s)}\mathbf{P}^{-1}\mathbf{A}_{\sigma(s)}^T\mathbf{C}_{i,\sigma(t)}^T\mathbf{P}\mathbf{e}(t)\right)ds \tag{22}$$

By using the Lyapunov–Razumikhin theorem for $\theta \in [-\max(\bar{\tau}_r), 0]$,[15,32] equation (22) can be expressed as $q > 1$:

$$\dot{V} \leqslant \mathbf{e}^T\left\{\mathbf{P}\left(\mathbf{A}_{\sigma(t)} + \sum_{i=1}^{N}\mathbf{B}_{i,\sigma(t)} + \sum_{r=1}^{\bar{m}}\mathbf{C}_{r,\sigma(t)}\right) + \left(\mathbf{A}_{\sigma(t)} + \sum_{i=1}^{N}\mathbf{B}_{i,\sigma(t)} + \sum_{r=1}^{\bar{m}}\mathbf{C}_{r,\sigma(t)}\right)^T\mathbf{P}\right\}\mathbf{e} + \\ + \mathbf{e}^T(t)\left(\sum_{i=1}^{N}q\mathbf{P}\Delta_i\right)\mathbf{e}(t) + \sum_{i=1}^{N}\int_{t-\Delta_i}^{t}\mathbf{e}^T(t)\mathbf{P}\mathbf{B}_{i,\sigma(t)}\mathbf{A}_{\sigma(s)}\mathbf{P}^{-1}\mathbf{A}_{\sigma(s)}^T\mathbf{B}_{i,\sigma(t)}^T\mathbf{P}\mathbf{e}(t)ds + \\ + \mathbf{e}^T(t)\left(\sum_{i=1}^{\bar{m}}q\mathbf{P}\bar{\tau}_i\right)\mathbf{e}(t) + \sum_{i=1}^{\bar{m}}\int_{t-\bar{\tau}_i}^{t}\mathbf{e}^T(t)\mathbf{P}\mathbf{C}_{i,\sigma(t)}\mathbf{A}_{\sigma(s)}\mathbf{P}^{-1}\mathbf{A}_{\sigma(s)}^T\mathbf{C}_{i,\sigma(t)}^T\mathbf{P}\mathbf{e}(t)ds \tag{23}$$

In previous (and similar) studies on time-varying networks, the Razumikhin theorem is not applied correctly.[17,23] To clarify the matter, consider the arbitrary switching matrix $\boldsymbol{\Xi}_{\sigma(t)}$. Since σ is a function of time, the expression $\int_{t-\bar{\tau}}^{t}\boldsymbol{\Xi}_{\sigma(s)}\mathbf{e}(s)ds = \boldsymbol{\Xi}_{\sigma}\int_{t-\bar{\tau}}^{t}\mathbf{e}(s)ds$ implies that in the time period $[t-\bar{\tau}, t]$ no switching actions occur. Since $\bar{\tau}$ is an arbitrary positive value and t is a free index, this assumption implies that the switching action will never happen in any time periods, which is a contradiction. To simplify equation (23), it is assumed that the percentage of the activity of jth subsystem is equal to α_{ij} in $[0, \Delta_i]$ and equal to β_{ij} in $[0, \bar{\tau}_i]$, in which $\sum_{j=1}^{n_s}\alpha_{ij} = 1$ and $\sum_{j=1}^{n_s}\beta_{ij} = 1$. If the kth subsystem is activated at time t ($\sigma(t) = k$), the equation (23) is written in the form of:

$$
\begin{aligned}
\dot{V} \leqslant\ & \mathbf{e}^T\left\{\mathbf{P}\left(\mathbf{A}_{\sigma(t)} + \sum_{i=1}^{N}\mathbf{B}_{i,\sigma(t)} + \sum_{r=1}^{\bar{m}}\mathbf{C}_{r,\sigma(t)}\right) + \left(\mathbf{A}_{\sigma(t)} + \sum_{i=1}^{N}\mathbf{B}_{i,\sigma(t)} + \sum_{r=1}^{\bar{m}}\mathbf{C}_{r,\sigma(t)}\right)^T\mathbf{P}\right\}\mathbf{e} + \\
& + \Delta_{\max}\mathbf{e}^T\left\{\sum_{i=1}^{N}\mathbf{PB}_{i,k}\left\{\sum_{j=1}^{n_s}\alpha_{ij}\mathbf{A}_j\mathbf{P}^{-1}\mathbf{A}_j^T\right\}\mathbf{B}_{i,k}^T\mathbf{P} + Nq\mathbf{P}\right\}\mathbf{e} + \\
& + \bar{\tau}_{\max}\mathbf{e}^T\left(\sum_{i=1}^{\bar{m}}\mathbf{PC}_{i,k}\left\{\sum_{j=1}^{n_s}\beta_{ij}\mathbf{A}_j\mathbf{P}^{-1}\mathbf{A}_j^T\right\}\mathbf{C}_{i,k}^T\mathbf{P} + \bar{m}q\mathbf{P}\right)\mathbf{e}
\end{aligned} \tag{24}
$$

where $\Delta_{\max} = \max\{\Delta_i, \mathrm{i} = 1, \ldots, N\}$ and $\bar{\tau}_{\max} = \max\{\bar{\tau}_r, \mathrm{r} = 1, \ldots, \bar{m}\}$. Subsequently, equation (24) can be written as:

$$
\dot{V} \leqslant -\mathbf{e}^T\mathbf{Q}_k\mathbf{e} + \Delta_{\max}p_1\mathbf{e}^T\mathbf{e} + \bar{\tau}_{\max}p_2\mathbf{e}^T\mathbf{e} \tag{25}
$$

where

$$
\begin{aligned}
p_1 &= \max_{k=1,\ldots,n_s}\left\{\max_{\substack{0\leqslant\alpha_{ij}\leqslant 1\\ \alpha_{i1}+\alpha_{i2}+\ldots+\alpha_{in_s}=1}}\lambda_{\max}\left\{\sum_{i=1}^{N}\mathbf{PB}_{i,k}\left\{\sum_{j=1}^{n_s}\alpha_{ij}\mathbf{A}_j\mathbf{P}^{-1}\mathbf{A}_j^T\right\}\mathbf{B}_{i,k}^T\mathbf{P} + Nq\mathbf{P}\right\}\right\} \\
p_2 &= \max_{k=1,\ldots,n_s}\left\{\max_{\substack{0\leqslant\beta_{ij}\leqslant 1\\ \beta_{i1}+\beta_{i2}+\ldots+\beta_{in_s}=1}}\lambda_{\max}\left\{\sum_{i=1}^{\bar{m}}\mathbf{PC}_{i,k}\left\{\sum_{j=1}^{n_s}\beta_{ij}\mathbf{A}_j\mathbf{P}^{-1}\mathbf{A}_j^T\right\}\mathbf{C}_{i,k}^T\mathbf{P} + \bar{m}q\mathbf{P}\right\}\right\}
\end{aligned} \tag{26}
$$

If the following conditions are met, $\dot{V}$ is negative definite:

$$
\bar{\tau}_{\max} < \lambda_{\min}(\mathbf{Q}_k)/p_2, \quad \Delta_{\max} < (\lambda_{\min}(\mathbf{Q}_k) - \tau_2 p_2)/p_1 \tag{27}
$$

Theorem 2. If the following conditions hold, the 1-D heterogeneous platoon of vehicles is internal stable under arbitrary switching.

(1) The lead vehicle is globally reachable in all subsystems.
(2) The following inequalities are satisfied:

$$
\begin{aligned}
& K < \frac{\lambda_{\min}(\mathbf{Z}_1)}{\lambda_{\max}(\mathbf{Z}_2)\lambda_{\max}(\mathbf{Z}_3)} \\
& \mathbf{Z}_1 = \begin{pmatrix} 2(D-1)\bar{\mathbf{I}} & D\bar{\mathbf{I}}^2 - (D-1)\bar{\mathbf{I}} - \mathbf{I} \\ D\bar{\mathbf{I}}^2 - (D-1)\bar{\mathbf{I}} - \mathbf{I} & 2\left(\bar{\mathbf{I}}^2 - \mathbf{I}\right) \end{pmatrix}, \\
& \mathbf{Z}_2 = \begin{pmatrix} \mathbf{I} & \bar{\mathbf{I}} \\ \bar{\mathbf{I}} & \bar{\mathbf{I}} \end{pmatrix}, \ \mathbf{Z}_3 = \bar{\mathbf{H}}_k\left(\bar{\mathbf{H}}_k + \bar{\mathbf{H}}_k^T\right)^{-1}\bar{\mathbf{H}}_k^T
\end{aligned}
$$

Proof. According to lemma 1, $\bar{\mathbf{H}} \succ \mathbf{0}$ if the lead vehicle is globally reachable in the platoon. Consider the following CLF

$$
V = \mathbf{e}^T\mathbf{P}\mathbf{e}, \quad \mathbf{P} = \begin{pmatrix} D\bar{\mathbf{I}} & \bar{\mathbf{I}} & \mathbf{I} \\ \bar{\mathbf{I}} & D\bar{\mathbf{I}} & \mathbf{I} \\ \mathbf{I} & \mathbf{I} & \bar{\mathbf{I}} \end{pmatrix}, \quad D > 2T_{\max} - 1 \tag{28}
$$

where $T_{\max} = \max\{T_i, i = 1, \ldots, N\}$. By taking the time derivative of V along equation (11) and by following the procedures similar to equations (16)–(25), $\dot{V}$ will be in the following form

$$
\begin{aligned}
& \dot{V} \leqslant -\mathbf{e}^T\bar{\mathbf{Q}}_k\mathbf{e} + \Delta_{\max}p_1\mathbf{e}^T\mathbf{e} + \bar{\tau}_{\max}p_2\mathbf{e}^T\mathbf{e}, \\
& \bar{\mathbf{Q}}_k = -\left(\mathbf{F}_k^T\mathbf{P} + \mathbf{PF}_k\right) \\
& \bar{\mathbf{Q}}_k = \begin{pmatrix} K\left(\bar{\mathbf{H}}_k + \bar{\mathbf{H}}_k^T\right) & K\bar{\mathbf{H}}_k^T & K\bar{\mathbf{I}}\bar{\mathbf{H}}_k^T \\ K\bar{\mathbf{H}}_k & 2(D-1)\bar{\mathbf{I}} & D\bar{\mathbf{I}}^2 - (D-1)\bar{\mathbf{I}} - \mathbf{I} \\ K\bar{\mathbf{H}}_k\bar{\mathbf{I}} & * & 2\left(\bar{\mathbf{I}}^2 - \mathbf{I}\right) \end{pmatrix}
\end{aligned} \tag{29}
$$

Exploiting lemmas 2 and 4, it is inferred that $\bar{\mathbf{Q}}_k \succ \mathbf{0}$ if:

$$
\begin{aligned}
& \begin{pmatrix} 2(D-1)\bar{\mathbf{I}} & D\bar{\mathbf{I}}^2 - (D-1)\bar{\mathbf{I}} - \mathbf{I} \\ D\bar{\mathbf{I}}^2 - (D-1)\bar{\mathbf{I}} - \mathbf{I} & 2\left(\bar{\mathbf{I}}^2 - \mathbf{I}\right) \end{pmatrix} \\
& - K\begin{pmatrix} \mathbf{I} & \bar{\mathbf{I}} \\ \bar{\mathbf{I}} & \bar{\mathbf{I}} \end{pmatrix} \otimes \bar{\mathbf{H}}_k\left(\bar{\mathbf{H}}_k + \bar{\mathbf{H}}_k^T\right)^{-1}\bar{\mathbf{H}}_k^T \succ \mathbf{0}
\end{aligned} \tag{30}
$$

which leads to $K < \frac{\lambda_{\min}(\mathbf{Z}_1)}{\lambda_{\max}(\mathbf{Z}_2)\lambda_{\max}(\mathbf{Z}_3)}$. Now, if the following conditions are met:

$$
\bar{\tau}_{\max} < \lambda_{\min}(\bar{\mathbf{Q}}_k)/p_2, \quad \Delta_{\max} < \left(\lambda_{\min}(\bar{\mathbf{Q}}_k) - \tau_2 p_2\right)/p_1 \tag{31}
$$

then $\dot{V}$ is negative definite.

Remark. As it is discussed in,[15] the Razumikhin-based theorems present small bounds for communication delay. Therefore, in the following, a Krasovskii-based theorem is presented which is less conservatism and present larger bound of delay.

Theorem 3. Under the following conditions, the switched linear system (11) is globally asymptotically stable under arbitrary switching.

(1) The lead vehicle is globally reachable in all subsystems.
(2) $\frac{D}{T_{\max}^2} - K\bar{\lambda}_R > 0,\ \underline{\lambda}_R\left[K\underline{\lambda}_R - \frac{D}{T_{\max}^2}\right]^2 - \frac{1}{T_{\min}^3}\underline{\lambda}_I^2 > 0$
(3) There exist symmetric matrices $\mathbf{P},\ \mathbf{Q}_i,\ \mathbf{S}_i,\ \mathbf{X}_i^\sigma = \begin{pmatrix} \mathbf{X}_{i,11}^\sigma & \mathbf{X}_{i,12}^\sigma \\ * & \mathbf{X}_{i,22}^\sigma \end{pmatrix} \succ 0,\ \mathbf{X}_{i,jk}^\sigma \in \Re^{N\times N}$ and arbitrary matrices $\mathbf{N}_{i,1}^\sigma, \mathbf{N}_{i,2}^\sigma,\quad i = 1,2,\ldots,N^2$, such that, the following expressions hold

$$\mathbf{P} \succ 0,\quad \mathbf{Q}_i \succ 0,\quad \mathbf{S}_i \succ 0 \tag{32}$$

$$\boldsymbol{\psi}_i^\sigma = \begin{pmatrix} \mathbf{X}_{i,11}^\sigma & \mathbf{X}_{i,12}^\sigma & \mathbf{N}_{i,1}^\sigma \\ * & \mathbf{X}_{i,22}^\sigma & \mathbf{N}_{i,2}^\sigma \\ * & * & \mathbf{S}_i \end{pmatrix} \succ 0,$$

$$\boldsymbol{\Phi}_\sigma = \begin{pmatrix} \boldsymbol{\Phi}_{1,1}^\sigma & \boldsymbol{\Phi}_{1,2}^\sigma & \cdots & \boldsymbol{\Phi}_{1,N^2+1}^\sigma \\ * & \boldsymbol{\Phi}_{2,2}^\sigma & \cdots & \boldsymbol{\Phi}_{2,N^2+1}^\sigma \\ \vdots & \vdots & \ddots & \vdots \\ * & \cdots & * & \boldsymbol{\Phi}_{N^2+1,N^2+1}^\sigma \end{pmatrix} \prec 0 \tag{33}$$

where

$$\begin{aligned}\boldsymbol{\Phi}_{1,1}^\sigma &= \mathbf{A}_\sigma^T\mathbf{P} + \mathbf{P}\mathbf{A}_\sigma \\ &+ \sum_{i=1}^{N^2}\mathbf{Q}_i + \sum_{i=1}^{N^2}\bar{\alpha}_i\mathbf{A}_\sigma^T\mathbf{S}_i\mathbf{A}_\sigma \\ &+ \sum_{i=1}^{N^2}\left(\mathbf{N}_{i,1}^\sigma + \mathbf{N}_{i,1}^{\sigma^T}\right) + \sum_{i=1}^{N^2}\bar{\alpha}_i\mathbf{X}_{i,11}^\sigma, \\ \boldsymbol{\Phi}_{i,i}^\sigma &\overset{(i\neq 1)}{=} -(1-\hat{\alpha}_i)\mathbf{Q}_{i-1} \\ &+ \sum_{j=1}^{N^2}\bar{\alpha}_j\bar{\mathbf{A}}_{i-1,\sigma}^T\mathbf{S}_j\bar{\mathbf{A}}_{i-1,\sigma} - \mathbf{N}_{i-1,2}^\sigma - \mathbf{N}_{i-1,2}^{\sigma^T} \\ &+ \bar{\alpha}_{i-1}\mathbf{X}_{i-1,22}^\sigma, \\ \boldsymbol{\Phi}_{1,j}^\sigma &\overset{(1\leq j)}{=} \mathbf{P}\bar{\mathbf{A}}_{j-1,\sigma} + \sum_{i=1}^{N^2}\bar{\alpha}_i\mathbf{A}_\sigma^T\mathbf{S}_i\bar{\mathbf{A}}_{j-1,\sigma} - \mathbf{N}_{j-1,1}^\sigma \\ &+ \mathbf{N}_{j-1,2}^{\sigma^T} + \bar{\alpha}_{j-1}\mathbf{X}_{j-1,12}^\sigma, \\ \boldsymbol{\Phi}_{i,j}^\sigma &\overset{(i<j,i\neq 1)}{=} \sum_{i=1}^{N^2}\bar{\alpha}_k\bar{\mathbf{A}}_{i-1,\sigma}^T\mathbf{S}_i\bar{\mathbf{A}}_{j-1,\sigma}.\end{aligned}$$

Also, $\bar{\mathbf{A}}_{i,\sigma} = \begin{cases}\mathbf{B}_{i,\sigma}, i = 1,\ldots,N \\ \mathbf{C}_{i-N,\sigma}, i = N+1,\ldots,N^2\end{cases},\quad 0 \leqslant \alpha_i(t) \leqslant \bar{\alpha}_i,$ $0 \leqslant \dot{\alpha}_i(t) \leqslant \hat{\alpha}_i \leqslant 1$, and $\alpha_i(t) = \begin{cases}\tau_i(t), i = 1,\ldots,N \\ \Delta_{i-N}, i = N+1,\ldots,N^2\end{cases}.$

Proof. The proof of conditions (1) and (2) is similar to theorem 1. In the continuance of the proof, consider the following common Lyapunov–Krasovskii function

$$\begin{aligned}V &= \boldsymbol{e}^T\mathbf{P}\boldsymbol{e} + \sum_{i=1}^{N^2}\int_{t-\alpha_i(t)}^{t}\boldsymbol{e}^T(s)\mathbf{Q}_i\boldsymbol{e}(s)ds \\ &+ \sum_{i=1}^{N^2}\int_{-\bar{\alpha}_i}^{0}\int_{t+\theta}^{t}\dot{\boldsymbol{e}}^T(s)\mathbf{S}_i\dot{\boldsymbol{e}}(s)dsd\theta\end{aligned} \tag{34}$$

The closed-loop dynamics of equation (11) can be written in the following form

$$\dot{\boldsymbol{e}} = \mathbf{A}_\sigma\boldsymbol{e} + \sum_{i=1}^{N^2}\bar{\mathbf{A}}_{i,\sigma}\boldsymbol{e}(t-\alpha_i(t)). \tag{35}$$

Taking time derivative of V along equation (35) leads to the following expression

$$\begin{aligned}\dot{V} &\leqslant \left[\mathbf{A}_\sigma\boldsymbol{e} + \sum_{i=1}^{N^2}\bar{\mathbf{A}}_{i,\sigma}\boldsymbol{e}_i\right]^T\mathbf{P}\boldsymbol{e} \\ &+ \boldsymbol{e}^T\mathbf{P}\left[\mathbf{A}_\sigma\boldsymbol{e} + \sum_{i=1}^{N^2}\bar{\mathbf{A}}_{i,\sigma}\boldsymbol{e}_i\right] \\ &+ \sum_{i=1}^{N^2}\boldsymbol{e}^T\mathbf{Q}_i\boldsymbol{e} - \sum_{i=1}^{N^2}(1-\hat{\alpha}_i)\boldsymbol{e}_i^T\mathbf{Q}_i\boldsymbol{e}_i + \\ &+ \sum_{i=1}^{N^2}\bar{\alpha}_i\left[\mathbf{A}_\sigma\boldsymbol{e} + \sum_{i=1}^{N^2}\bar{\mathbf{A}}_{i,\sigma}\boldsymbol{e}_i\right]^T \\ &\mathbf{S}_i\left[\mathbf{A}_\sigma\boldsymbol{e} + \sum_{i=1}^{N^2}\bar{\mathbf{A}}_{i,\sigma}\boldsymbol{e}_i\right] \\ &- \sum_{i=1}^{N^2}\int_{-\bar{\alpha}_i}^{0}\dot{\boldsymbol{e}}^T(t+\theta)\mathbf{S}_i\dot{\boldsymbol{e}}(t+\theta)d\theta\end{aligned} \tag{36}$$

where $\boldsymbol{e}_i = \boldsymbol{e}(t-\alpha_i(t))$. By adding the following obvious terms to the right hand side of equation (36),

$$\begin{aligned}&2\left[\boldsymbol{e}^T\mathbf{N}_{i,1}^\sigma + \boldsymbol{e}_i^T\mathbf{N}_{i,2}^\sigma\right]\left[\boldsymbol{e} - \boldsymbol{e}_i - \int_{-\bar{\alpha}_i}^{0}\dot{\boldsymbol{e}}(t+\theta)d\theta\right] = 0, \\ &\bar{\alpha}_i\boldsymbol{\delta}_i^T\mathbf{X}_i^\sigma\boldsymbol{\delta}_i - \int_{-\bar{\alpha}_i}^{0}\boldsymbol{\delta}_i^T\mathbf{X}_i^\sigma\boldsymbol{\delta}_i d\theta = 0\end{aligned} \tag{37}$$

where $\boldsymbol{\delta}_i = [\boldsymbol{e}^T, \boldsymbol{e}_i^T]^T$, and doing some simplifications, equation (36) can be written as follows:

$$\dot{V} \leqslant \boldsymbol{\gamma}_0^T\boldsymbol{\Phi}_\sigma\boldsymbol{\gamma}_0 - \sum_{i=1}^{N^2}\int_{-\bar{\alpha}_i}^{0}\boldsymbol{\gamma}_i^T\boldsymbol{\psi}_i^\sigma\boldsymbol{\gamma}_i d\theta \tag{38}$$

where $\boldsymbol{\gamma}_0 = [\boldsymbol{e}^T, \boldsymbol{e}_1^T, \ldots, \boldsymbol{e}_{N^2}^T]^T$ and $\boldsymbol{\gamma}_i = [\boldsymbol{e}^T, \boldsymbol{e}_i^T, \dot{\boldsymbol{e}}^T]^T$. If the conditions of equations (32) and (33) are satisfied, $\dot{V}$ is negative-definite. As a result, all subsystems of equation (11) are globally asymptotically stable. Moreover, V is a CLF; thus stability in switching instants is assured.

Table 1 presents a comparison between theorems 1, 2, and 3 from different points of view.

Table 1. Comparison between presented theorems.

Aspect	**Upper bound of communication delay**	**Content of calculations**	**Robustness against lag**	**Simplicity of Lyapunov function**	**Complexity of approach**
Theorem	3 > 1 > 2	3 > 1 > 2	3 > 1 > 2	2 > 1 > 3	1 > 2 > 3

Table 2. Parameters used in simulation studies.

Parameters	Description	Value
K	Gain of controller	2.7
D	Gain of controller	4.1
$\bar{\tau}_{ij}(s)$	Communication delay between vehicles	$0.18\vert\sin t\vert$
$h_i(s)$	Constant time headway	0.8
$D_{\min}(m)$	Minimum displacement	5

String stability

As mentioned previously, the string stability of a vehicular platoon (either homogeneous or heterogeneous) with MPF topology under time-varying delay has not been solved in previous studies. For MPF topology, the closed-loop dynamics of ith vehicle will be in the following form:

$$T_i\dddot{e}_i + \ddot{e}_i = -D\dot{e}_i(t-\Delta_i) - Ke_i(t-\Delta_i) - K\sum_{j=i-m-1}^{i-1}\left[e_i(t-\Delta_i) - e_j(t-\bar{\tau}_{ij})\right] \tag{39}$$

Equation (39) can be rewritten as follows:

$$T_i\dddot{e}_i + \ddot{e}_i = -D\dot{e}_i(t-\Delta_i) - K(m+1)e_i(t-\Delta_i) + K\sum_{j=i-m-1}^{i-1} e_j(t-\bar{\tau}_{ij}) \tag{40}$$

Theorem 4. Under the following condition, the string stability of system of equation (11) is assured under m-predecessors following topology.

$$D \geqslant \sqrt{2K(m+1)} \tag{41}$$

Proof. Taking the Laplace transform of both sides of equation (40) will result in

$$\left(T_is^3 + s^2\right)E_i(s) = -Dse^{-\Delta_i s}E_i(s) - K(m+1)e^{-\Delta_i s}E_i(s) + K\sum_{j=i-m-1}^{i-1} e^{-\tau_{ij}s}E_j(s) \tag{42}$$

By simplifying equation (42), the following equality is obtained:

$$E_i(s) = \frac{K}{T_is^3 + s^2 + [Ds + K(m+1)]e^{-\Delta_i s}}\sum_{j=i-m-1}^{i-1} e^{-\tau_{ij}s}E_j(s) \tag{43}$$

By considering that $\left|E_j(j\omega)\right| \leqslant E_{\max}$, from equation (43), we can conclude that

$$|E_i(j\omega)| \leqslant |G_i(j\omega)|E_{\max}, \quad G_i(s) = \frac{N_i(s)}{D_i(s)} = \frac{mK\sum_{j=i-m-1}^{i-1} e^{-\tau_{ij}s}}{T_is^3 + s^2 + [Ds + K(m+1)]e^{-\Delta_i s}} \tag{44}$$

So that if $|D_i(j\omega)|^2 - |N_i(j\omega)|^2 \geqslant 0,$ the string stability is assured. By performing some algebraic calculation, the following inequality is obtained

$$\begin{aligned} &T_i^2\omega^6 + \omega^4 + D^2\omega^2 - 2DT_i\omega^4\cos\Delta_i\omega \\ &+ 2KT_i(m+1)\omega^3\sin\Delta_i\omega \\ &- 2D\omega^3\sin\Delta_i\omega - 2K(m+1)\omega^2\cos\Delta_i\omega \\ &+ (m+1)^2K^2 - m^2K^2 \geqslant 0 \end{aligned} \tag{45}$$

According to

$$\begin{aligned} &\forall\delta \geqslant 0 : \sin\delta \leqslant \delta \rightarrow -\sin\delta \geqslant -\delta,\ \sin\delta \geqslant -\delta, \\ &\cos\delta \leqslant 1 \rightarrow -\cos\delta \geqslant -1, \end{aligned} \tag{46}$$

equation (45) will be simplified as follows

$$\begin{aligned} &T_i^2\omega^6 + [1 - 2D\Delta_i - 2DT_i - 2KT_i\Delta_i(m+1)]\omega^4 \\ &+ \left[D^2 - 2K(m+1)\right]\omega^2 + (m+1)^2K^2 - m^2K^2 \geqslant 0 \end{aligned} \tag{47}$$

Since the spacing errors have most of their energy in the area with low frequencies, this area is determinative in string stability analysis.[35] So that, if the following condition hold, the string stability is assured, i.e.

$$D^2 - 2K(m+1) \geqslant 0 \Rightarrow D \geqslant \sqrt{2K(m+1)}$$

Safety (collision avoidance) during emergency braking

In a string stable platoon, the spacing error of the first vehicle is larger than other vehicles. If we have $e_1(t) \leqslant L^*$, where L^* is a safe distance, then the platoon is safe during emergency braking. Therefore, we can express that

$$\frac{e_1(t)}{a_0(t)} \leqslant \frac{L^*}{a_{0,\max}} \Rightarrow \frac{E_1(s)}{A_0(s)} \leqslant \frac{L^*}{a_{0,\max}s} \Rightarrow \left|\frac{E_1(j\omega)}{A_0(j\omega)}\right| \leqslant \frac{L^*}{a_{0,\max}\omega} \tag{48}$$

where $a_0(t)$ and $a_{0,\max}$ are deceleration and max deceleration of lead vehicle during sudden braking.

Theorem 5. Under the following conditions, the collision avoidance is assured during emergency braking.

$$\begin{aligned} &1 + 2KT_1\Delta_1 - 2D(T_1 + \Delta_1) - a_{0,\max}^2T_1^2/L^{*2} \geqslant 0, \\ &D^2 - a_{0,\max}^2/L^{*2} \geqslant 0 \end{aligned} \tag{49}$$

Proof. In emergency braking, $a_0 \neq 0$. Therefore, for $i = 1$, equation (10) can be written in the following form

$$T_1\dddot{e}_i + \ddot{e}_1 = -D\dot{e}_1(t-\Delta_1) - Ke_1(t-\Delta_1) + T_1\dot{a}_0 + a_0 \tag{50}$$

Taking the Laplace transform of both sides of equation (50) will result in

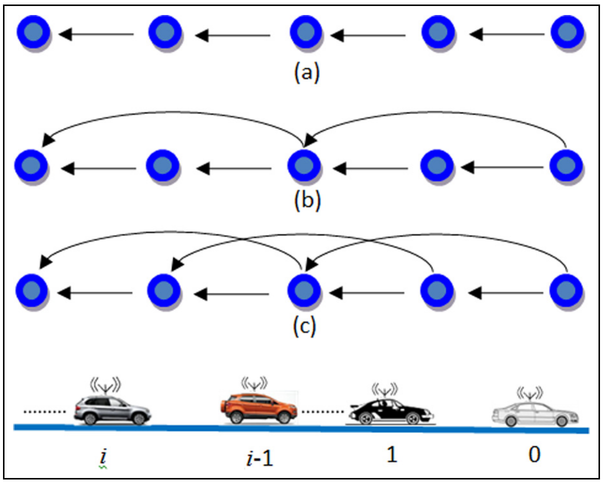

Figure 3. Switching topology of a heterogeneous platoon.

$$\left|\frac{E_1(j\omega)}{A_0(j\omega)}\right| = \frac{|j\omega T_1 + 1|}{\left|-j\omega^3 T_1^3 - \omega^2 + (j\omega D + K)e^{-j\omega\Delta_1}\right|} \quad (51)$$

Combining equations (48) and (51) and employing equation (46), it can be easily shown that under the conditions of equation (49), the heterogeneous platoon of vehicles is safe during emergency braking.

Simulation study

A platoon of six vehicles consisting of three different kinds of vehicles is considered, as shown in Figure 3. It is assumed that the communication topology of each vehicle varies between leader predecessor following (LPF) and leader two-predecessors following (LTPF) schemes. Three different topologies are considered as shown in Figure 3.

In order to show the effect of velocity of platoon on inter-vehicle spacing, in all scenarios, the spacing error is defined as $\delta_i = x_{i-1} - x_i - L_{i-1}$. The relation between tracking error and spacing error is as follows $\delta_i = e_i + h_i v_0 + D_{\min}$. Under the control input of equation (7), e_i converges to zero and subsequently, δ_i converges to $h_i v_0 + D_{\min}$.

Scenario 1. In this scenario, the performance of string stability of platoon is studied. The control parameters are presented in Table 2. The engine time constant, parasitic delay and length of vehicles are considered as follows:

$$T_1 = 0.1s, T_2 = 0.11s, T_3 = 0.07s, T_4 = 0.12s, T_5 = 0.08s$$
$$\Delta_1 = 0.08s, \Delta_2 = 0.1s, \Delta_3 = 0.11s, \Delta_4 = 0.14s, \Delta_5 = 0.09s$$
$$L_1 = 4m, L_2 = 4.1m, L_3 = 3.8m, L_4 = 4.2m, L_5 = 3.9m$$

Figure 4 shows the velocity tracking and Figure 5 depicts the performance of string stability of platoon.

According to Figure 5, in the acceleration and deceleration time period of lead vehicle's motion, the amplitude of spacing error decreases along the platoon indicating the string stability.

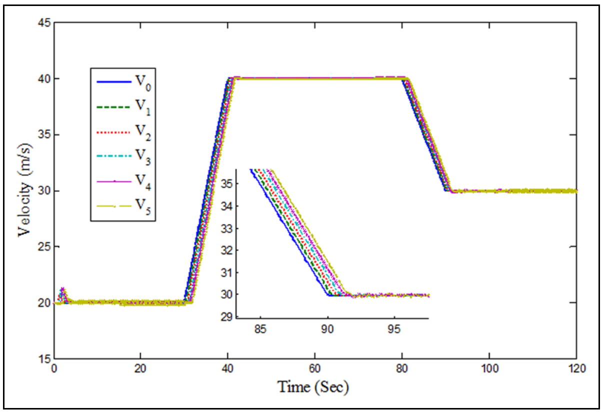

Figure 4. Velocity of vehicles in the first scenario.

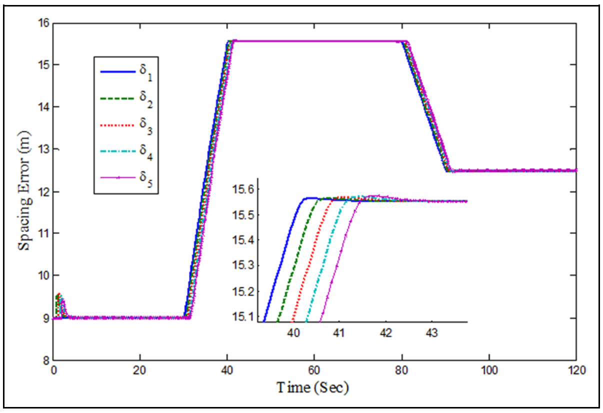

Figure 5. Spacing error for the first scenario.

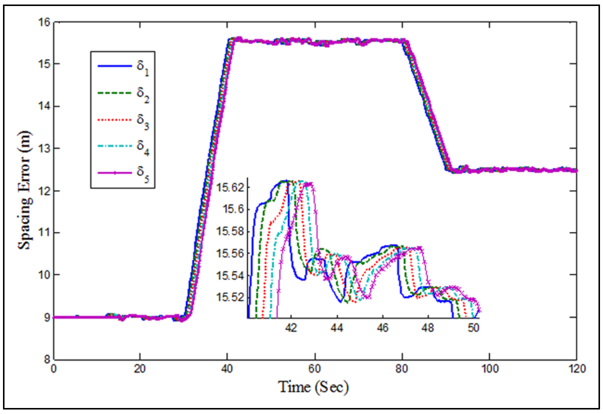

Figure 6. Spacing error for the second scenario.

Scenario 2. In this scenario, the performance of spacing error is studied in presence of noise on transmitted signals. A noise signal with the amplitude ± 0.6 and sampling time $T = 0.05$ *Sec* is applied to all transmitted signals between vehicles. Figure 6 shows the spacing error in this scenario. According to this figure, the control algorithm of equation (7) is robust against noise and guarantees both string and internal stability.

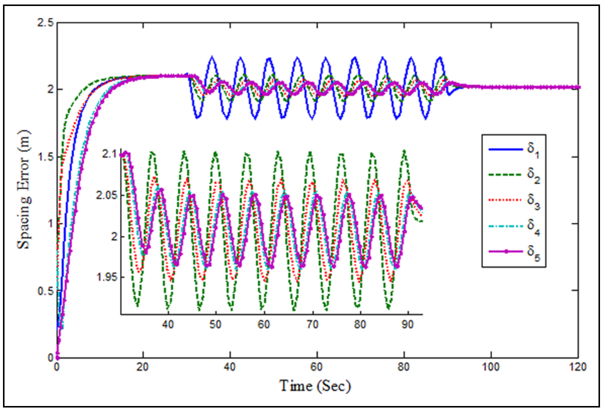

Figure 7. Spacing error in presence of disturbance.

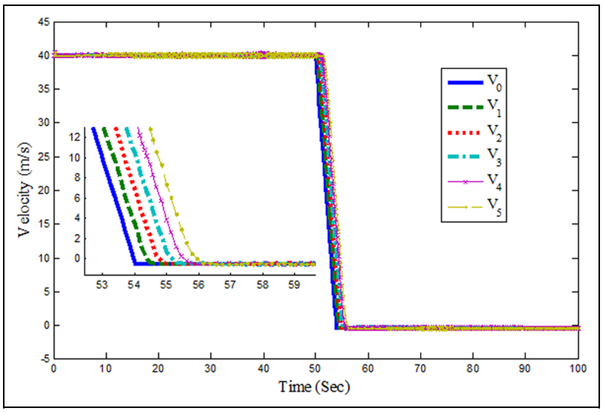

Figure 8. Velocity of vehicles in emergency braking.

Scenario 3. This scenario studies the influence of external disturbance on string stability of platoon. For this purpose, the disturbance signal $d(t) = \begin{cases} 1.23\sin(0.96t), & t > 30, t < 90 \\ 0, & otherwise \end{cases}$. is applied to lead vehicle's motion. According to Figure 7, the platoon is string stable against disturbance signal.

Scenario 4. In this scenario, the safety of the platoon in an emergency braking maneuver is studied. Figure 8 shows the velocity of vehicles. As this figure indicates, at $t = 50$s the emergency barking occurs. In this scenario, in addition to parameters described in Table 2, it is assumed that $a_{0,\max} = -10m/s^2$ and $L^* = 1.3m$. Figure 8 depicts the velocity of vehicles in this scenario and Figure 9 shows the spacing error during emergency braking. According to these figures, the collision avoidance is guaranteed during emergency braking.

Conclusion

The third-order safe consensus of longitudinal heterogeneous vehicular platoons is considered in this paper. The network topology of platoon is considered time-varying. Both communication and parasitic delays are considered in control design and stability analyses. A centralized neighbor based linear control law based on

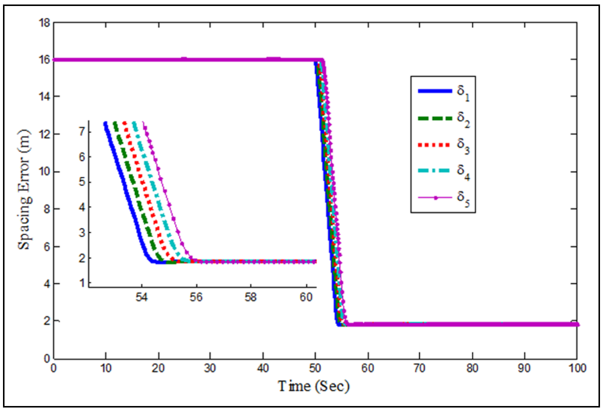

Figure 9. Spacing error in emergency braking.

MPF topology is considered for each vehicle. Some new approaches are presented to perform internal stability analysis of the vehicular platoons in presence of delay and time-varying network topology. Afterwards, a new theorem is presented which introduces a necessary condition on control parameters to guarantee the string stability for MPF topology. Moreover, necessary conditions on control parameters assuring safety during emergency braking are derived. Several simulation studies are rendered to illustrate the effectiveness of the proposed approaches.

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Appendix

List of symbols and abbreviations

$\mathbf{A}$:	Adjacency matrix of followers,
$\mathbf{B}$:	Adjacency matrix of leader,
$\mathbf{B}_M$:	Bilharz matrix,
CSS:	Constant spacing strategy,
CTHS:	Constant time headway strategy,
c_i :	Lower level control,
D:	Gain of controller,
$D_{\min}$:	Minimum displacement,
h_i :	Constant time headway,
ITS:	Intelligent transportation system,
K:	Gain of controller,
$\mathbf{N}_{ij}^{\sigma}$:	Arbitrary matrices,
n_s :	Number of subsystems,
N:	Number of vehicles,
$\bar{\mathbf{P}}, \mathbf{P}, \mathbf{Q}_i, \mathbf{S}_i, \mathbf{X}_i^{\sigma}$:	Positive definite matrices,
x_i :	Position of vehicle i,
$\dot{x}_i, v_i$:	Velocity of vehicle i,
$\ddot{x}_i, a_i$:	Acceleration of vehicle i,
x_i^d :	Desired position,
T:	Engine time constant,
$\sigma(t)$:	Switching signal,
$\lambda_{i,k}$:	ith eigenvalue of $\mathbf{H}_k$,
$\zeta_{i,k}$:	ith characteristic equation,
$\tau(t)$:	Communication delay,
$\bar{\tau}_i(t)$:	Total time delay,
$\bar{\tau}_i$:	Maximum total time delay,
Δ :	Parasitic delay,
δ_i :	Spacing error.