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Pricing and two-dimensional warranty policy of multi-products with online and offline channels using a value-at-risk approach

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ABSTRACT

Nowadays, manufacturers often sell different products (models of a product) through both online and offline channels. To face challenges such as legislation or competition, manufacturers often need to provide warranties for their products. On the one hand, customers are more attracted to goods covered with a warranty; on the other hand, warranties generate additional costs for manufacturers. This study investigates a case that a manufacturer sells through online and offline channels and offers a two-dimensional warranty policy including a warranty-age and a warranty-usage package for sold products through online channel. Because different models share several components, failures of all manufacturer's portfolio products are statistically dependent. A model to optimize both the pricing and warranty policies is proposed. Given that the claim rates for warranties are stochastic, the value-at-risk approach is implemented to solve the optimization problem. Furthermore, the covariance between warranty claims associated with different products is computed through a copula. The findings indicate the importance of considering the covariance among different models claims when optimal warranty policy is offered to customers along with the proper pricing strategy in an online channel.

1. Introduction

Often sales revenue and costs of after-sales services affect the profit of a company. Since customers concern about the affordability and reliability of products when making a purchase decision, a good warranty policy and pricing will influence and eventually maximize the profit of the company. Thus, companies need to implement related marketing tools, like pricing, advertising, warranty, etc. to affect customers' decisions (Xie, 2017). Typically, customers judge the reliability of products through warranty length because a warranty forces the company to be responsible for all failures during the warranty contract. Also, the product's price has always been the most important factor to be considered in purchasing. Therefore, a company must study the joint effect of warranty policy and price to understand the economic consequence (Mas-Colell, Whinston, & Green, 1995). Warranty, as an expensive after-sale service, plays a significant part in the sale promotion. According to the customer behaviors study, the warranty can improve a company's image and attract more customers (Boulding & Kirmani, 1993; Purohit & Srivastava, 2001).

Many companies often implement a 2-dimensional (2D) warranty policy in which warranty age and warranty usage are limited simultaneously. The warranty policies typically possess both age and usage dimensions for capital-intensive industries, including engines, automobiles, heavy equipment. Implementation of 2D warranty has successful in these industries (Blischke, Karim, & Murthy, 2011).

By increasing the use of the Internet, many companies sell their product in two different channels, (i.e., online and offline channels). The Internet is used by a growing number of customers for different shopping purposes. Ernst and Young (2015) surveyed about 7000 consumers and about 70 CEOs in 12 developed countries and discovered that multi-channel companies are more popular among customers. According to the features of each sale channel, different policies and different pricing can be used to sell products.

On the other hand, companies often produce multiple products or different models of a product by using similar components. Note that in this study, we use products and different models of a product interchangeably. With the increasing personalization of products by customers, most industries, such as automobiles industry, have turned to produce different models of a base product. Using similar components causes dependency between the products' failure rates, which means there is statistical dependency among the number of different products' warranty claims. For example, the sunroof of several models of automobiles may be similar or several types of automobiles may use the same kind of engine.

Therefore, motivated by the important topics mentioned above, this study proposes an integrated model for optimal pricing of multi-

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products by considering 2D warranty policy and dual sale channels. Since a company's profit depends on its warranty policy and product pricing in each channel, the proposed model seeks to maximize the company's profits by optimally determining product price per each channel, warranty age and warranty usage.

Regarding the uncertainty on the number of failure (warranty) claims (NFC) over time, the value at risk (VaR) method is implemented to solve this model. Under a specified probability, VaR is a measure of the risk-of-loss for investments and determines how much a set of investments might lose. Usually, the financial industries employ VaR to assess the number of assets needed to cover possible losses. Also, for modeling the dependency among products, the copulas as a more flexible tool than simple tools such as covariance estimations are used. Therefore, according to VaR's definition, the optimal price of each product is determined to maximize the profits of a company under a VaR constraint by considering the covariance among the products.

The remainder of this study is arranged as follows: in Section 2, the related works are surveyed and the contributions of this study are highlighted; in Section 3, the proposed model and related concepts are developed; in Section 4, two solution methodologies namely mean-variance and VaR are proposed to solve the problem; finally, in Section 5, the study is concluded and the important results and future opportunity are mentioned.

2. Literature review

The published works related to this study consists of two main streams: the pricing-warranty problem and the online/offline sales channel problem.

Research in the first category seeks to investigate the various warranty-related impact, like warranty packages on the companies' profit. Glickman and Berger (1976) studied the effect of the warranty period on companies' pricing and profits for the first time. They showed the benefits of both pricing and warranty policy on the company's profit. Later, Murthy (1990) proposed a model that consists of the product price, warranty policy, and reliability. Wu, Lin, and Chou (2006) extended Glickman and Berger's model by considering two different strategies: fixed and dynamic pricing. Huang, Liu, and Murthy (2007) further developed the idea in Glickman and Berger (1976) by integrating R&D and production costs. By categorizing customers according to their maintenance data, Huang, Huang, and Ho (2017) provided a pricing model for a 2D warranty policy. Hereby, it was showed that customer categorization is important for an effective warranty pricing mechanism and marketing strategy. Also, regarding the maximization of the total profits of a company, Aggrawal, Anand, Singh, and Singh (2014) developed a 2D innovation diffusion model for determining the price and warranty length. Yazdian, Shahanaghi, and Makui (2016) introduced a model to optimize the price and length of warranty applying linear and nonlinear demand functions. Xie (2017) proposed a model to optimize the profit of a new product by considering a 2D warranty policy. In this model, warranty usage limitation, product price, and warranty age limitation, as three decision variables, were considered. Also, they assumed that the arrival rate of warranty claims follows the Nonhomogeneous Poisson Process.

Zhang, He, He, and Dai (2019) considered trade-off between warranty cost and boosted demand for optimizing a 2D warranty policy. For this, they proposed a demand function based on attractiveness index to model the customer's demand function and validated their model in the automobile market. Shang, Si, Sun, and Jin (2018) incorporated condition-based maintenance to optimize the warranty policy, by which the product reliability in the warranty period can be tracked. Through this model, the warranty period, sale price, and replacement threshold are determined, which showed that the replacement threshold should be equal and below the failure threshold for monopoly and competitive markets, respectively. Cheong, Cheong, Zhang, and Zhang (2019) proposed a dynamic optimization model the price under limitation of both warranty length and usage. Accordingly, they proposed a nonlinear optimal control problem to solve their problem and developed a new sales function to characterize the joint influence of pricing and warranty policy on customers' demands. Tang, Li, Li, Liu, and Huang (2019) analyzed the pricing and warranty decisions using a Stackelberg game for a two-period closed-loop supply chain. Through it, they identified the conditions under which warranty for remanufactured products is offered and its effects.

Since in the real world, products are often made up of several components, researchers have investigated pricing under this situation. Matis, Jayaraman, and Rangan (2008) proposed the optimal length of pro-rated warranty and price for a product which consists of multi-component under various repair options of the components. Ahmadi (2016) developed a model for optimizing the replacement problem of complicated multi-part systems and seeking the optimal operating period, which should consider the tradeoffs between the incomes and the associated costs. Chen, Lo, and Weng (2017) presented a total profit maximization model for each item through optimizing the warranty length and production run length. As the recent study in this area, Luo and Wu (2017) considered dependencies among the arrival rates of warranty claims of different products which use common components. They used a VaR approach to optimize the warranty period and price of a portfolio of products. Luo and Wu (2018) developed a mean-variance approach to determine the pricing and warranty policy for a portfolio of products and used copulas for depicting the dependence among the warranty claims of different products.

In the second category of studies, with the increasing popularity of the Internet and application of dual sales channels, researchers surveyed the impact of the online channel. In this regard, many studies examined the behavior of customers on the Internet or investigated the optimal choice of sales channels. The impact of the transaction cost on choosing a channel was examined by Chintagunta, Chu, and Cebollada (2012), who discover it as a vital factor in the grocery retailers. Melis, Campo, Breugelmans, and Lamey (2015) investigated the creation of a channel for online sales and the choice of sales channels by customers. By assuming that customers want to maximize their utility, they indicated that customers tend to buy from online stores which have been previously purchased with the offline channel. Wang, Lin, Tai, and Fan (2016) examined the impact of characteristics of sales channels on customers' channel selection. Also, Feng, Li, Xu, and Deng (2019) developed a model for implementing a trade-in programme through retail and direct channels simultaneously and finding the optimal price and trade-in policies.

Chu, Arce-Urriza, Cebollada-Calvo, and Chintagunta (2010) examined the behavior of customer purchases in multi-channel stores and realized that loyalty and brand reputation in online sales have a greater impact on the sales of the offline than the price of the product. Arce-Urriza, Cebollada, and Tarira (2017) investigated the effects of price promotion in the brand selection of the online and offline channels for a retailer, and they found that price promotion has more impact on the offline channel. Dan, Zhang, and Zhou (2017) investigated the warranty policy in a two-channel supply chain system that includes a producer and a retailer, but with the only producer providing warranty service. In their study, additional free services were included to attract more customers, and subsequently, the optimal warranty policy and competition outcomes were compared with and without such an assumption.

Based on our literature review, there are too few studies that have investigated covariance among different components/models of products for a company establishing a product portfolio. Since the products are made by common components, the failure rates of these products are related. Accordingly, considering the covariance among the components and products can significantly improve warranty policy and consequently optimal pricing and profits. In addition, due to the growing use of the Internet, multiple sales channels are used in many companies, i.e. online and offline channels with respectively dedicated customers. Also, many companies implement a 2D warranty policy based on the features of their products by which the warranty policy can be determined more profitably. In short, it is important to consider all these attributes to understand their effect on pricing and warranty policy and consequently the profit of a company. Due to the research gap in the literature, the current study is dedicated to integrate two sale channels, 2D warranty policy, and multi-products with the consideration of covariance among them and find the optimal pricing as well as warranty length and usage by a Value-at-Risk (VaR) approach. Therefore, the main novelties of this study are as follows:

- Developing a VaR approach for simultaneously optimizing the 2D warranty policy and pricing in a dual-channel sales manufacturer for the first time
- Considering the covariance among the different products which are produced by the manufacturer and modeling it using a Copula
- Analyzing the optimal solutions under different scenarios through numerical examples

3. Problem description and modeling

3.1. Problem definition

Consider a company that produces a portfolio of products. These products consist of some common components. The company sells its products through two channels, i.e. products can be sold directly in the online channel and alternatively by retailers in the offline channel. The company assigns a policy that the warranty is only granted to online sales. The company has a 2-dimensional non-renewing free replacement warranty (2D NFRW) policy. Accordingly, the company is obligated to repair or replace the failed components of a product without any cost until the termination of the warranty term. When time or usage of a product reaches a specified limit level, the warranty term would be terminated. Suppose that the time of repair is negligible and products are new at the time of sale, and all the failures in the warranty are rectified by minimal repair (Bernard & Vanduffel, 2015; He, Zhang, Zhang, & He, 2017). Therefore, NFC can be modeled by the non-homogeneous Poisson process (NHPP) (He et al., 2017). Also, it is assumed that there is no statistical dependency between the NFC and claim costs.

There are two critical factors that affect the sales volume of a product. The price of the product is negatively related to its sales volume. Whereas warranty strategy positively increases the sales volume. More specifically, by increasing the time or usage of limit in a warranty strategy, the sales volume of the product increases (Chen et al., 2017). There are different models representing the above relation between the volume of sales and pricing and warranty strategy (e.g. (Lin, Wang, & Chin, 2009; Yazdian et al., 2016; He et al., 2017; Huang, Gau, & Ho, 2015) proposed linear models and (Huang et al., 2007; Xie, Liao, & Zhu, 2014) proposed nonlinear models). This study combines the models of (Huang et al., 2015; Yazdian et al., 2016) and introduces a new model for 2-D warranty strategy and two sales channels, proposed as Eqs. (1)–(3). All notations are summarized in Table 1.

$$V_{k,o} = C_{k,o} - \beta_{k,o}P_{k,o} + \lambda_k T_k + \gamma_k U_k + \alpha_f \beta_{k,f} P_{k,f} \tag{1}$$

$$V_{k,f} = C_{k,f} - \beta_{k,f}P_{k,f} - \alpha_o(\lambda_k T_k + \gamma_k U_k - \beta_{k,o}P_{k,o}) \tag{2}$$

Accordingly, the total profit of the product $k$calculated as follows.

$$W_k = V_{k,o}P_{k,o} + V_{k,f}P_{k,f} - O_k(T_k, U_k) \tag{3}$$

The Standard probability distributions such as lognormal, Gamma can be used for modeling the cost of failure (Klugman, Panjer, & Willmot, 2012; McNeil, Frey, & Embrechts, 2005). It is assumed that $X_{k,j}$ follows a log-normal distribution, which is subexponential. The aggregated warranty cost of product k is subexponential. Therefore, $X_{k,j}$ and $O_k(T_k, U_k)$ are tail equivalent (Bee, 2017; Bernard & Vanduffel, 2015). To make our paper concise, we implemented the log–normal distribution. For more information on tail equivalent, please refer to (Bee, 2017). Also, assuming the costs of failures to be independent, we can calculate $O_k(T_k, U_k)$ as:

Table 1
Notations and description.

Notation	Description
$V_{k,i}$	Sale volume of product k in channel $i \in \{o, f, T\}$ o for online, f for offline and T for total
$P_{k,i}$	Price of product k in channel $i \in \{o, f\}$ o for online and f for offline
$X_{k,j}$	The cost of the $j'th$ claim of product k
W_k	The total profit of product k
$C_{k,i}(>0)$	Market size of product k in channel $i \in \{o, f\}$ o for online and f for offline
T_k	Warranty time of product k in online channel
U_k	Warranty usage of product k in online channel
TL_k	The minimum threshold of the warranty usage of product k
UL_k	The minimum threshold of the warranty time of product k
$O_k(T_k, U_k)$	The aggregated warranty cost of product k
$N_k(T_k, U_k)$	The number of failures over the warranty contract
$\Lambda_k(T_k, U_k)$	Intensity function of 2-D warranty policy which follows NHPP
f_{O_k}	The PDF of $O_k(T_k, U_k)$
F_{O_k}	The CDF of $O_k(T_k, U_k)$
$\rho_{X_k}(t, u)$	The characteristic function (CF) of $X_{k,j}$
$\rho_{O_k}(t, u)$	The CF of $O_k(T_k, U_k)$
$\psi_k(O)$	The probability generation function of $N_k(T_k, U_k)$
$\beta_{k,i}$	Price elasticity of product k in the channel $i \in \{o, f\}$ o for online and f for offline
λ_k	Warranty time elasticity of product k in online channel
γ_k	Warranty usage elasticity of product k in online channel
$\partial_{k,1}, \partial_{k,2}$	The shape (law power) parameters of bivariate Weibull process
α_i	Percent of channel i selling that goes to another channel because of price or warranty strategy in that channel, $i \in \{o, f\}$ o for online and $f$for offline

$$O_k(T_k, U_k) = \sum_{j=1}^{N_k(T_k,U_k)} X_{k,j} \tag{4}$$

It is assumed that the suitable process of two-dimensional warranty claims model is the bivariate Weibull process (Lu & Bhattacharyya, 1990). This follows the same thought as in Huang et al. (2015), where a bivariate Weibull process is used for 2-D NFRW. Therefore, we have:

$$\Lambda_k(T_k, U_k) = \left(\frac{T_k}{\theta_{k,1}}\right)^{\partial_{k,1}}\left(\frac{U_k}{\theta_{k,2}}\right)^{\partial_{k,2}} \tag{5}$$

where $\theta_{k,i} > 0$ for $i = 1, 2$. The distribution function of $N_k(T_k, U_k)$ is $\frac{(V_{k,o}\Lambda_k(T_k,U_k))^n}{n!}e^{-V_{k,o}\Lambda_k(T_k,U_k)}$. The cost of the $j'th$ claim of product $k$is shown by $X_{k,j}$ which follows a log-normal distribution with its mean as μ_k and variance σ_k^2. With these assumptions, the expected value (E) and variance (Var) of $O_k(T_k, U_k)$ can be expressed as:

$$E(O_k(T_k, U_k)) = E[N_k(T_k, U_k)]E[X_k] = V_{k,o}\Lambda_k(T_k, U_k)\mu_k \tag{6}$$

$$\begin{aligned} Var(O_k(T_k, U_k)) &= E[N_k(T_k, U_k)]Var[X_k] + Var[N_k(T_k, U_k)]E[X_k]^2 \\ &= V_{k,o}\Lambda_k(T_k, U_k)(Var[X_k] + E[X_k]^2) = V_{k,o}\Lambda_k(T_k, U_k)(\sigma_k^2 + \mu_k^2) \end{aligned} \tag{7}$$

Denote that $\rho_{X_k}(t, u)$is equal to $\int_{-\infty}^{+\infty} f_{X_k}(x)e^{i'(t+u)x}dx$, where $i'$is a unit imaginary number. Also, $\psi_k(O)$is equal to $\sum_{n=0}^{+\infty} o^n p_{k,n}$, where $p_{k,n} = \Pr(N_k(T_k, U_k) = n)$. Based on Levy–Khintchine formula, it can be obtained that:

$$\rho_{S_k}(t, u) = \sum_{n=0}^{+\infty} (\rho_{X_k}(t, u))^n p_{k,n} = e^{V_{k,o}\Lambda_k(T_k,U_k)(\rho_{X_k}(t,u)-1)} \tag{8}$$

Then, the density function of $O_k(T_k, U_k)$, which is positive, can be calculated using inverse two-dimensional Fourier transform as follows:

$$f_{O_k(T_k,U_k)}(x,y)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\rho_{S_k}(t,u)e^{-i(tx+uy)}dtdu$$
$$=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}e^{V_{k,o}\Lambda_k(T_k,U_k)(\rho_{X_k}(t,u)-1)}e^{-i(tx+uy)}dtdu \quad (9)$$

The cumulative function of $O_k(T_k, U_k)$is calculated as:

$$F_{O_k(T_k,U_k)}(x,y)=\int_0^y\int_0^x f_{O_k(T_k,U_k)}(t,u)dtdu \quad (10)$$

In practice, calculating PDF and CDF by the Fourier transform requires great computational power to overcome the overflow and underflow errors. The well-known method is employing an approximating function to bypass direct computation of Eqs. (9) and (10). According to Bee (2017), for each X_k with subexponential function, the compound $O_k(T_k, U_k)$ inherits the subexponentiality attribute. In current study, it is assumed that X_k follows the log-normal distribution, which is subexponential. Thus, $O_k(T_k, U_k)$ is subexponential too. Accordingly, the log-normal distribution for the approximation of $O_k(T_k, U_k)$ is used here. The characteristics (μ_{1-n},σ^2_{1-n}) of log-normal distribution are as follows:

$$\mu_{1-n}=E(O_k(T_k,U_k))=V_{k,o}\Lambda_k(T_k,U_k)\mu_k \quad (11)$$

$$\sigma^2_{1-n}=Var(O_k(T_k,U_k))=V_{k,o}\Lambda_k(T_k,U_k)(\sigma_k^2+\mu_k^2) \quad (12)$$

3.2. Single and multiple product scenarios

In a single product system, the profit is calculated by Eq. (3). The expected value and variance of profit can be derived as:

$$E[W_k]=E[V_{k,o}P_{k,o}+V_{k,f}P_{k,f}-O_k(T_k,U_k)]$$
$$=V_{k,o}(P_{k,o}-\Lambda_k(T_k,U_k)\mu_k)+V_{k,f}P_{k,f} \quad (13)$$

$$Var(W_k)=Var(O_k(T_k,U_k))=V_{k,o}\Lambda_k(T_k,U_k)(\sigma_k^2+\mu_k^2) \quad (14)$$

$$F_{W_k}(z)=P(W_k\leqslant z)=P(V_{k,o}P_{k,o}+V_{k,f}P_{k,f}-O_k(T_k,U_k)\leqslant z)=$$
$$P(O_k(T_k,U_k)\geqslant V_{k,o}P_{k,o}+V_{k,f}P_{k,f}-z)$$
$$=1-F_{O_k(T_k,U_k)}(V_{k,o}P_{k,o}+V_{k,f}P_{k,f}-z) \quad (15)$$

Whereas in a company with multiple products, a portfolio of products exists. In this situation, the company's profit is equal to the total profit of the products (e.g. Luo and Wu (2017)). Then its profit is calculated as follows:

$$W_T=\sum_{k=1}^{N}W_k=\sum_{k=1}^{N}(V_{k,o}P_{k,o}+V_{k,f}P_{k,f}-O_k(T_k,U_k)) \quad (16)$$

$$F_{W_T}(x,y)=P(W_T\leqslant z)=F^{(N)}(z) \quad (17)$$

where $F^{(N)}(z)$ is the N-fold convolution of the distribution of W_k. Based on Eq. (11), the expected value of the product portfolio becomes:

$$E[W_T]=\sum_{k=1}^{N}(V_{k,o}(P_{k,o}-\Lambda_k(T_k,U_k)\mu_k)+V_{k,f}P_{k,f}) \quad (18)$$

The arrival process of warranty claims of the products could be correlated since the company produces different products consisting of some components. Hence the variance of a portfolio of products depends on the correlation between them. Assuming that the correlations are linear, the variance calculation of W_T is expressed as follows:

$$Var(W_T)=I^TQI \quad (19)$$

where $I^T=[1, 1, ...,1]$ and Q is covariance matrix with the expression:

$$Q=\begin{bmatrix} Var(O_1(T_1,U_1)) & Cov(O_1(T_1,U_1),O_2(T_2,U_2)) & \cdots & Cov(O_1(T_1,U_1),O_n(T_n,U_n)) \\ Cov(O_2(T_2,U_2),O_1(T_1,U_1)) & Cov(O_2(T_2,U_2),O_2(T_2,U_2)) & \cdots & Cov(O_2(T_2,U_2),O_n(T_n,U_n)) \\ \cdots & \cdots & \cdots & \cdots \\ Cov(O_n(T_n,U_n),O_1(T_1,U_1)) & Cov(O_n(T_n,U_n),O_2(T_2,U_2)) & \cdots & Var(O_n(T_n,U_n)) \end{bmatrix} \quad (20)$$

where $Cov(O_k(T_k,U_k),O_i(T_i,U_i))=\rho_{k,i}\sqrt{Var(O_k(T_k,U_k))Var(O_i(T_i,U_i))}$ and $\rho_{k,i}$ is the Pearson correlation coefficient. There exist different types of dependency between the warranty costs of different products (e.g., tail-dependence and rank correlation). Using the metrics that just measure a linear correlation is inappropriate in the case where the relationship of the variables is nonlinear. For this reason, a powerful tool named copula is used to model the dependence between the products. The copula parameters, mainly the Archimedean copula parameters, are used as substitutes for measuring relationships in the covariance matrix in multi-products optimization, with the relationship to be linear or non-linear (Boubaker & Sghaier, 2013).

4. Solution methodology

In this study, we use the mean-risk optimization for maximizing the expected profit under the constraint of considering a risk level. Eqs. (13) and (16) express the objective function for one product scenario and a portfolio of products scenario, respectively. By considering Eqs. (1), (2), (5) and (13), we have the profit of one product scenario as:

$$E[W_k]=(C_{k,o}-\beta_{k,o}P_{k,o}+\lambda_kT_k+\gamma_kU_k+\alpha_f\beta_{k,f}P_{k,f})(P_{k,o}$$
$$-(\frac{T_k}{\theta_{k,1}})^{\partial_{k,1}}(\frac{U_k}{\theta_{k,2}})^{\partial_{k,2}}\mu_k)+$$
$$(C_{k,f}-\beta_{k,f}P_{k,f}-\alpha_o(\lambda_kT_k+\gamma_kU_k-\beta_{k,o}P_{k,o}))P_{k,f} \quad (21)$$

In this case, we may have in total 4 decision variables, namely $P_{k,o}$, $P_{k,f}$, T_k, and U_k. We now introduce Proposition 1 based on the above expression.

Proposition 1. *Depending on whether $P_{k,o}$, $P_{k,f}$, T_k, and U_k are decision variables or given ones, we have:*

If more than one variables are decision variables, there is no global optimum solution for $E[W_k]$.

If one of the above variables is decision variable and others are given, there is a global optimum solution for $E[W_k]$.

The proving of this proposition is illustrated in Appendix A.

Also, the profit in multi-product scenario is calculated as follows:

$$E[W_T]=\sum[(C_{k,o}-\beta_{k,o}P_{k,o}+\lambda_kT_k+\gamma_kU_k+\alpha_f\beta_{k,f}P_{k,f})(P_{k,o}$$
$$-(\frac{T_k}{\theta_{k,1}})^{\beta_1}(\frac{U_k}{\theta_{k,2}})^{\beta_2}\mu_k)+$$
$$(C_{k,f}-\beta_{k,f}P_{k,f}-\alpha_o(\lambda_kT_k+\gamma_kU_k-\beta_{k,f}P_{k,f}))P_{k,f}] \quad (22)$$

Since no dependency is considered between the prices and sales volumes in this study, Proposition 1 is also valid in the multi-product scenario. The dependence amongst the warranty claims of the products is reflected in the constraints of optimizations. The risk is defined as a position's future value instability because of unknown issues (Artzner, Delbaen, Eber, & Heath, 1999). It describes a position where a portfolio is subjected to vulnerabilities and enforces damages to the business (Babaei, Sepehri, & Babaei, 2015). It is measured by the variable's variance in some original models for selecting the portfolio. However, Value-at-Risk (VaR) and Conditional-Value-at-Risk (CVaR) are then used by researchers, and these approaches indicate a more valid concept of risk (Babaei et al., 2015). VaR is one of the most popular downside risk measures, which indicates a more valid concept of risk. The variance as a risk measure is normally applied under the assumption that the correlation is linear, which is not forced in the VaR and CVaR theories (Luo & Wu, 2017). Although, maximum possible loss cannot be determined only by VAR and VaR gets difficult to calculate with large portfolios, its advantages outweigh its disadvantages, which can be seen through its popularity in financial management (Babaei et al., 2015). So that in this study, by considering profit maximization under the mean-risk, the mean-variance and VaR metrics are used to measure the risk.

4.1. Optimizing mean-risk value in one product scenario

In this scenario, three different types of optimization are conducted to maximize the product's expected profit with a given risk.

4.1.1. Mean-Variance model

In this model, the profit maximization has a constraint level of variance is the goal. The optimization model is expressed in Eqs. (23)–(27).

$$MAX\,E(W_k) = V_{k,o}(P_{k,o} - (\frac{T_k}{\theta_{k,1}})^{\partial_{k,1}}(\frac{U_k}{\theta_{k,2}})^{\partial_{k,2}}\mu_k) + V_{k,f}P_{k,f} \tag{23}$$

Subject to

$$V_{k,o}(\frac{T_k}{\theta_{k,1}})^{\partial_{k,1}}(\frac{U_k}{\theta_{k,2}})^{\partial_{k,2}}(\sigma_k^2 + \mu_k^2) \leqslant \zeta \tag{24}$$

$$V_{k,o} = C_{k,o} - \beta_{k,o}P_{k,o} + \lambda_k T_k + \gamma_k U_k + \alpha_f\beta_{k,f}P_{k,f} \geqslant 0 \tag{25}$$

$$V_{k,f} = C_{k,f} - \beta_{k,f}P_{k,f} - \alpha_o(\lambda_k T_k + \gamma_k U_k - \beta_{k,o}P_{k,o}) \geqslant 0 \tag{26}$$

$$P_{k,f},\ P_{k,o} \geqslant 0,\ T_k \geqslant TL_k,\ U_k \geqslant UL_k \tag{27}$$

Where Eq. (23) indicates our objective, and Eq. (24) ensures that variance does not exceed an acceptable level. Eqs. (25) and (26) ensure that the sales volume should be non-negative. And Eq. (27) ensures that the price of the product should be greater than or equal to zero, and meanwhile warranty time and warranty usage should be greater than or equal to their minimum threshold. According to Proposition 1, one variable can act as the decision variable. For example, if $P_{k,o}$ is the decision variable, then Eq. (24) can rewrite as follows:

$$P_{k,o} \geqslant \frac{1}{\beta_{k,o}}\left(C_{k,o} + \lambda_k T_k + \gamma_k U_k + \alpha_f\beta_{k,f}P_{k,f} - \frac{\zeta}{\left(\frac{T_k}{\theta_{k,1}}\right)^{\partial_{k,1}}\left(\frac{U_k}{\theta_{k,2}}\right)^{\partial_{k,2}}(\sigma_k^2 + \mu_k^2)}\right) = P_{k,o}^1 \tag{28}$$

Therefore, to satisfy the constraint (24), $P_{k,o}$ must be equal or greater than $P_{k,o}^1$. Using the derivative of Eq. (21) with respect to $P_{k,o}$, we have the optimal $P_{k,o}$ for global maximum computed as:

$$P_{k,o} = \frac{C_{k,o} + \lambda_k T_k + \gamma_k U_k + \alpha_f\beta_{k,f}P_{k,f} - \beta_{k,o}\left(\frac{T_k}{\theta_{k,1}}\right)^{\partial_{k,1}}\left(\frac{U_k}{\theta_{k,2}}\right)^{\partial_{k,2}}\mu_k}{2\beta_{k,o} - \alpha_o\beta_{k,o}P_{k,f}} = P_{k,o}^2 \tag{29}$$

Regarding the value of $P_{k,o}^1$and $P_{k,o}^2$, the optimal point can be obtained since the model is only constrained by Eq. (24), therefore if $P_{k,o}^1 \leqslant P_{k,o}^2$, $P_{k,o}$ is not constrained by Eq. (24) and then the global maximum of expected profit is at $P_{k,o}^2$, otherwise, it is at $P_{k,o}^1$. This procedure can be applied to all other variables. Then, Proposition 2 is expressed.

Proposition 2. *The global maximum of expected profit under a predefined variance level exists when one variable is the decision variable and others are known and the power-law parameters ($\partial_{k,1}$, $\partial_{k,2}$) is greater than 1.*

The proof of Proposition 2 is illustrated in Appendix B.

4.1.2. Mean value at risk model

The θ − quantile of a distribution, where θ is a proposed confidence level, is called VaR. Here, $VaR(W_k)$ is defined as the amount of lost profit at the $(1-\theta)$ level. So, $(1-\theta)$is the probability that $VaR(W_k)$of profit of product k may lose. Then, it can be calculated as follows:

$$VaR_\theta(W_k) = F_{W_k}^{-1}(\theta) \tag{30}$$

Using Eq. (15), we have the expression re-written as:

$$VaR_\theta(W_k) = V_{k,o}P_{k,o} + V_{k,f}P_{k,f} - F_{O_k(T_k,U_k)}^{-1}(1-\theta) \tag{31}$$

Then, it is possible to develop the VaR model as follows:

$$MAX\,E(W_k) = V_{k,o}(P_{k,o} - (\frac{T_k}{\theta_{k,1}})^{\partial_{k,1}}(\frac{U_k}{\theta_{k,2}})^{\partial_{k,2}}\mu_k) + V_{k,f}P_{k,f} \tag{32}$$

Subject to

$$VaR_\theta(W_k) = V_{k,o}P_{k,o} + V_{k,f}P_{k,f} - F_{O_k(T_k,U_k)}^{-1}(1-\theta) \leqslant \zeta \tag{33}$$

$$V_{k,o} \geqslant 0,\ V_{k,f} \geqslant 0,\ P_{k,f},\ P_{k,o} \geqslant 0,\ T_k \geqslant TL_k,\ U_k \geqslant UL_k \tag{34}$$

Where Eq. (33) indicates that the $VaR(W_k)$ of the product at the $(1-\theta)$ confidence level must be less than ζ, $(\zeta \geqslant 0)$. Now, suppose that we want to find the optimal $P_{k,o}$. In this case, all other variables are known, then $V_{k,o}P_{k,o}$ in $VaR_\theta(W_k)$ is a parabola with a negative coefficient on the quadratic term (Luo & Wu, 2017). $F_{O_k(T_k,U_k)}^{-1}(1-\theta)$ obtained from the warranty cost's distribution. As we suppose other variables are known, the expected value and variance of $O_k(T_k, U_k)$ depend only on $P_{k,o}$, and furthermore these two values monotonously reduce with an increase of $P_{k,o}$. Also, $F_{O_k(T_k,U_k)}^{-1}(1-\theta)$ monotonously reduces with an increase of$P_{k,o}$.

Due to Propositions 1 and 2, and above discussion, the following proposition can be obtained.

Proposition 3. *The global maximum of expected profit under a VaR level exists if one variable is the decision variable and others are known.*

4.2. Optimizing mean-risk value in multi-product scenario

In this section, we will develop mean-variance and mean VaR frameworks for maximizing expected profit in the multiple product scenario. Suppose that a company produces N products having common components. In such cases, it is reasonable to assume the dependency among warranty claim arrival processes. By considering this statistical dependency, it is possible to model this problem in mean-variance and mean VaR frameworks. In the following, we investigate this problem.

4.2.1. Mean-Variance model

With a portfolio of products, we optimize the expected profit of N products with defined variance as Eqs. (35) and (36).

$$MAX\,E[W_T] = \sum_{k=1}^{N}(V_{k,o}(P_{k,o} - (\frac{T_k}{\theta_{k,1}})^{\partial_{k,1}}(\frac{U_k}{\theta_{k,2}})^{\partial_{k,2}}\mu_k) + V_{k,f}P_{k,f}) \tag{35}$$

Subject to

$$Var(W_T) = \sum_{k=1}^{N}Var(W_k) + 2\sum_{k=1}^{N}\sum_{\substack{i=1\\ i\neq k}}^{N}Cov(O_k(T_k, U_k), O_i(T_i, U_i)) \leqslant \zeta \tag{36}$$

where $Cov(O_k(T_k, U_k), O_i(T_i, U_i)) = \rho_{k,i}\sqrt{V_{k,o}V_{i,o}(\frac{T_k}{\theta_{k,1}})^{\partial_{k,1}}(\frac{T_i}{\theta_{i,1}})^{\partial_{k,1}}(\frac{U_k}{\theta_{k,2}})^{\partial_{k,2}}(\frac{U_i}{\theta_{i,2}})^{\partial_{i,2}}(\sigma_k^2 + \mu_k^2)(\sigma_i^2 + \mu_i^2)}$,
and $\rho_{k,i}$ is Pearson correlation coefficient which is used to measure dependency among warranty claim of products. Using Proposition 2, the following proposition can be obtained.

Proposition 4. *The global maximum of expected total profit under a predefined variance level exists if one variable is the decision variable and others are known.*

4.2.2. Mean VaR model

The following model is created for optimization of the expected total profit of N products under mean VaR model:

$$MAX\,E[W_T] = \sum_{k=1}^{N} (V_{k,o}(P_{k,o} - (\frac{T_k}{\theta_{k,1}})^{\partial_{k,1}}(\frac{U_k}{\theta_{k,2}})^{\partial_{k,2}}\mu_k) + V_{k,f}P_{k,f}) \quad (37)$$

Subject to

$$VaR_{\theta}(W_T) = F^{-1,(N)}(1-\theta) \leqslant \zeta \quad (38)$$

Although the interpretation and definition of the VaR are easy and understandable, the calculation of VaR of a portfolio of products is not easy (Babaei et al., 2015). There are common methods to calculate VaR in finance including historical or stochastic simulation and the variance-covariance methods. In practice, it is possible and suitable to calculate VaR using simulation. The variance-covariance assumes that risk factors follow normal distributions, which makes it not appropriate in this case (Luo & Wu, 2017). Here, stochastic simulation is applied. One of the exact and efficient simulation methods in the probability theory is copula which is used to formalize the dependence structures between a portfolio of products, as we need in this study. This method enables us to compute the optimal solution of our VaR model.

Copula is used to construct multivariate distributions and formalize the random variables' dependency structures. This approach can be applied whether the variables are continuous or not (Sklar, 1959). It is feasible to rewrite each cumulative distribution of random variables using its marginal distribution and a copula with which the dependence structure among them is expressed. Recently, considerable attention has attracted the study of copulas, both from theoretical and practical viewpoints. Any cumulative distribution function of a random vector can be written in terms of marginal distribution functions and a copula that describes the dependence structure between the variables. This feature of Copula makes it an extremely powerful tool in statistical applications. Both Wu (2014) and Luo and Wu (2017) have implemented this method to model warranty claims of a portfolio of products.

Consider $Z = (Z_1, Z_1, ..., Z_n)$ as a random vector of variables that its CDF is $C(z_1, z_1, ..., z_n) = P(Z_1 \leqslant z_1, Z_2 \leqslant z_2, ..., Z_n \leqslant z_n)$ and its marginal distribution function of each variable is $Md_k(z_k) = P(Z_k \leqslant z_k)$ for $k = 1, 2, ..., n$. It can be proved that $C(z_1, z_1, ..., z_n) = Co(Md_1(z_1), Md_2(z_2), ..., Md_n(z_n))$ where $Co(.)$ is a copula (Sklar, 1959). Then using this method, we have:

$$C(z_1, z_1, ..., z_n) = Co\left(F^{(N_1)}(z_1), F^{(N_2)}(z_2), ..., F^{(N_k)}(z_k), ..., F^{(N_n)}(z_n)\right) \quad (39)$$

$$c(z_1, z_1, ..., z_n) = co\left(F^{(N_1)}(z_1), F^{(N_2)}(z_2), ..., F^{(N_n)}(z_n)\right)\prod_{k=1}^{N} f^{(N_k)}(z_k) \quad (40)$$

where $C(z_1, z_1, ..., z_n)$ and $c(z_1, z_1, ..., z_n)$ are the joint distribution (JD) of NFC of the product and its density. $F^{(N_1)}(z_1)$ and $f^{(N_k)}(z_k)$ are CDF and PDF of NFC of a product. And the density of copula $Co(.)$ is $co(.)$. Finally, $C(z_1, z_1, ..., z_n)$ can be simulated using Eqs. (9), (10), (39), and (40). Since the calculation of Eqs. (9) and (10) is difficult and complex, the log-normal distribution is used to approximate them. Then, Eqs. (11) and (12) are used instead of Eqs. (9) and (10).

Consider $F_{W_T}(z)$ as the total profit distribution of the portfolio, where $W_T = \sum_{k=1}^{N} W_k$. Using a copula-based model, enable us to estimate VaR through simulation. Computing F is mainly a numerical problem (Bernard & Vanduffel, 2015).

Then using Proposition 3, the following proposition can be obtained.

Proposition 5. *The global maximum of expected total profit under a predefined VaR level exists if one variable is the decision variable and others are known.*

In practice, a variety of copula families exists. A proper copula can be built or chosen in two steps. First, according to the physical condition of the products or the features of empirical operating, the form of marginal distribution, and the tail-dependence, etc., a fitting copula family can be chosen (Luo & Wu, 2017). For instance, an Elliptical family copula form can be chosen for linear dependency; an Archimedean family copula form can be chosen for data in which rank correlation is observed. Second, the performance of the chosen copulas must be examined by some tools like Bayesian Information Criterion or mean squared errors (Luo & Wu, 2017).

5. Numerical examples with a case study

As highlighted in Section 1, the 2D-warranty is applied widely in the automotive industry where products share different components and are sold through both online and offline channels. To test the validity of our model, we collected data from an automotive company producing 5 types of vehicles. An ABC survey on the failures of different models shows that about 90% of the failures are correlated to 7 components. The warranty data (claims' date and cost) are collected for estimated the related parameters and other parameters such as price, warranty time, and warranty usage elasticity of products are estimated by experts. To validate the assumptions used in this model, a simulation model is implemented to check the fitness of real data to the model. For example, the simulation model generated warranty claims using the NHPP and then generated NFC are tested against real NFC. Therefore, the model assumptions are checked and fitted to the real data. Table 2 shows the components used in each model. For example, components AA, BB, DD, and FF are used for producing model 1001 while components BB, CC, DD, and GG are used for producing model 1002. The failures are rectified by minimal repair and arrival of warranty claims is NHPPs with cumulative failure intensity as shown in Eq. (5). The parameters' values associated with each component are summarized in Table 3. For example, the claim of model 1001 is $(\frac{T_k}{370})^{1.02}(\frac{U_k}{220})^{1.03} + (\frac{T_k}{620})^{1.04}(\frac{U_k}{240})^{1.02} + (\frac{T_k}{520})^{1.01}(\frac{U_k}{280})^{1.05} + (\frac{T_k}{450})^{1.02}(\frac{U_k}{450})^{1.03}$. Furthermore, the sales information is reported in Table 4, with notations explained in our model. We also have $T_k = 720$ and $U_k = 400$ days for all models. Finally, the offline prices of each model ($P_{k,f}$) are displayed in Table 4. Using the data collected in Tables 3 and 4, we aim at find how the pricing of the online products can influence and finally maximize profits. This investigation can provide an empirical evidence to support our theoretical developments.

Before going deep, we solved and obtained the optimum price and warranty policy for product model 1001 by considering only offline sale channel, only online sale channel, both channels together (the study's case). For this, the related parameters of sales channels are considered equal (i.e., $C_{k,o} = C_{k,f} = 1000$, $\beta_{k,o} = \beta_{k,f} = 0.20$). If only one channel without warranty policy (offline channel is our case) is considered. By deriving Eq. (3), after removing online channel related parts, the optimum price is 2500 and the profit is 1250000. If only one channel with warranty policy (offline channel is this case) is considered. By solving

Table 2
Characteristics of models.

Model	Components						
	AA	BB	CC	DD	EE	FF	GG
1001	✓	✓		✓		✓	
1002		✓	✓	✓			✓
1003	✓		✓		✓	✓	✓
1004	✓	✓			✓	✓	✓
1005	✓		✓	✓	✓		

Table 3
Parameters' value for each component.

Component	$\theta_{k,1}$	$\partial_{k,1}$	$\theta_{k,2}$	$\partial_{k,2}$	Cost of each claim
AA	370	1.02	220	1.03	100
BB	620	1.04	240	1.02	80
CC	480	1.06	400	1.01	90
DD	520	1.01	280	1.04	120
EE	550	1.01	300	1.05	100
FF	450	1.05	450	1.03	50
GG	600	1.04	340	1.01	60

Eqs. (28) and (29), after removing offline channel related parts, the optimum price is 2750 and the profit is 1801085. Finally, considering both channels leads to a 2,181,950 profit under $C_{k,o} = C_{k,f} = 500$, $\beta_{k,o} = \beta_{k,f} = 0.20$, and $P_{k,f} = 2700$ values. This simple analysis shows that it is preferred to implement both channels.

To better investigate the covariance between different models, we conduct an empirical analysis under two situations: single-product and multi-product scenario.

5.1. Single-product scenario

In this section, we deal with a single product scenario while dealing with two cases. First, we assume that there is no restriction on $Var(W_k)$, therefore we can use Eq. (29) to determine the optimal online price $P_{k,o}$ and the optimal manufacturer's profit $E[W_k]$. Second, we assume that there is a restriction (ζ) on $Var(W_k)$, therefore we use Eq. (28) rather than Eq. (29). Also, if manufacturers seek to make a decision on other variables, the same procedure is followed for determining the optimal value.

5.1.1. Mean-Variance model

First when there is no restriction on $Var(W_k)$. By using the equations in Sections 3.1, 3.2, and 4.1.1, the expected profit and the related variance on each model are computed as follows:

$$E[W_1] = 324P_{k,o} - (0.2P_{k,o} - 1449.1)(P_{k,o} - 811.5116) - 11772 \quad (41)$$

$$Var(W_1) = 111450000 - 15382P_{k,o} \quad (42)$$

$$E[W_2] = 437.5P_{k,o} - (P_{k,o} - 542.5418)(0.25P_{k,o} - 1098.8) + 1114600 \quad (43)$$

$$Var(W_2) = 48413000 - 11015P_{k,o} \quad (44)$$

$$E[W_3] = 180P_{k,o} - (0.15P_{k,o} - 1625.2)(P_{k,o} - 740.3222) + 1463760 \quad (45)$$

$$Var(W_3) = 95864000 - 8847.9P_{k,o} \quad (46)$$

$$E[W_4] = 384P_{k,o} - (0.2P_{k,o} - 928)(P_{k,o} - 876.9144) - 80640 \quad (47)$$

$$Var(W_4) = 74172000 - 15985P_{k,o} \quad (48)$$

$$E[W_5] = 140P_{k,o} - (0.1P_{k,o} - 1108)(P_{k,o} - 617.5376) - 347200 \quad (49)$$

$$Var(W_5) = 48151000 - 4345.7P_{k,o} \quad (50)$$

We depict the relationship between $P_{k,o}$ and the manufacturer's profit for model 1001 in Fig. 1. The pattern of the profit functions of other models is similar, and therefore we will not include in the paper. All results are summarized in Table 5.

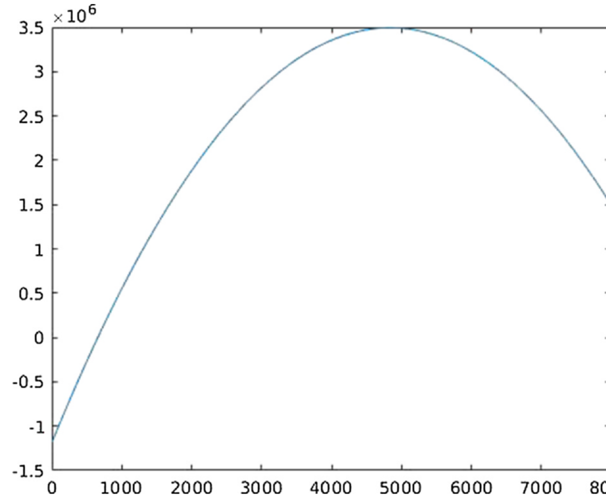

Fig. 1. Relationship between total profit and online price of product model 1001.

Table 5
Optimal $P^2_{k,o}$ and $E[W_k]$ using the mean–variance model.

Model (k)	$P^2_{k,o}$	$E[W_k]$	$Var(W_k)$
1001	4838.5	3,494,500	37,023,000
1002	3343.9	3,313,800	11,580,000
1003	6387.5	6,380,600	39,348,000
1004	3718.5	1,871,000	14,731,000
1005	6548.8	3,257,200	19,691,000

As can be seen from Fig. 1, there exists an optimal warranty length that maximizes the manufacturer's profits. As can be seen, the form of Fig. 1 graph is because of the quadratic form of the objective function.

5.1.2. Mean VaR model

Now, we assume that there is a restriction on $Var(W_k)$. Specifically, we use the mean-VaR model by assuming a confidence level of 0.05. The $VaR_{0.05}(W_k)$ for each model of the product is computed in Eqs. (51)–(55).

$$VaR_{0.05}(W_1) = 324\mathrm{P}_{k,o} - \mathrm{P}_{k,o}(0.2\mathrm{P}_{k,o} - 1449.1) - 11772 - F^{-1}_{O_1(T_1,U_1)}(0.95) \quad (51)$$

$$VaR_{0.05}(W_2) = 437.5\mathrm{P}_{k,o} - \mathrm{P}_{k,o}(0.25\mathrm{P}_{k,o} - 1098.8) + 1114600 - F^{-1}_{O_2(T_2,U_2)}(0.95) \quad (52)$$

$$VaR_{0.05}(W_3) = 180\mathrm{P}_{k,o} - \mathrm{P}_{k,o}(0.15\mathrm{P}_{k,o} - 1625.2) + 1463760 - F^{-1}_{O_3(T_3,U_3)}(0.95) \quad (53)$$

$$VaR_{0.05}(W_4) = 384\mathrm{P}_{k,o} - \mathrm{P}_{k,o}(0.2\mathrm{P}_{k,o} - 928) - 80640 - F^{-1}_{O_4(T_4,U_4)}(0.95) \quad (54)$$

Table 4
Parameters' value for each vehicle.

Model (k)	$C_{k,o}$	$C_{k,f}$	$\beta_{k,o}$	$\beta_{k,f}$	λ_k	γ_k	α_o	α_f	μ_k	σ_k	$P_{k,f}$
1001	1000	500	0.20	0.15	0.13	0.18	0.60	0.70	89.36	21.99	2700
1002	800	800	0.25	0.10	0.14	0.12	0.70	0.60	78.16	15.44	2500
1003	1200	1000	0.15	0.15	0.16	0.10	0.40	0.60	75.25	18.25	3000
1004	400	600	0.20	0.20	0.20	0.12	0.80	0.70	85.40	22.15	2400
1005	700	500	0.10	0.20	0.10	0.14	0.50	0.50	68.80	10.40	2800

Table 6
Characteristics of log-normal distribution for approximating $F^{-1}_{O_k(T_k,U_k)}(0.95)$.

Model (k)	$\mu_{\text{l-n}}$	$\sigma^2_{\text{l-n}}$	$F^{-1}_{O_k(T_k,U_k)}(0.95)$
1001	$1176000 - 162.3023P_{1,o}$	$111450000 - 15382P_{1,o}$	$1.65\sqrt{111450000-15382P_{1,o}} + 1176000 - 162.3023P_{1,o}$
1002	$596140 - 135.6354P_{2,o}$	$48413000 - 11015P_{2,o}$	$1.65\sqrt{48413000-11015P_{2,o}} + 596140 - 135.6354P_{2,o}$
1003	$1203200 - 111.0483P_{3,o}$	$95864000 - 8847.9P_{3,o}$	$1.65\sqrt{95864000-8847.9P_{3,o}} + 1203200-111.0483P_{3,o}$
1004	$813780 - 175.3829P_{4,o}$	$74172000 - 15985P_{4,o}$	$1.65\sqrt{74172000-15985P_{4,o}} + 813780-175.3829P_{4,o}$
1005	$684230 - 61.7538P_{5,o}$	$48151000 - 4345.7P_{5,o}$	$1.65\sqrt{48151000-4345.7P_{5,o}} + 684230-61.7538P_{5,o}$

$$VaR_{0.05}(W_5) = 120P_{k,o} - P_{k,o}(0.1P_{k,o} - 1068) - 105600 - F^{-1}_{O_5(T_5,U_5)}(0.95) \quad (55)$$

As we already mentioned in Section 3.1, we cannot obtain closed forms for the CDF of $O_k(T_k, U_k)$and $F^{-1}_{O_k(T_k,U_k)}(0.95)$. Consequently, we use a log-normal distribution to approximate the distribution of $F^{-1}_{O_k(T_k,U_k)}(0.95)$. The results of the log-normal distribution for approximating $F^{-1}_{O_k(T_k,U_k)}(0.95)$ are displayed in Table 6, in which mean and variance are derived from Eqs. (11)–(12). Using these equations along with Eqs. (32)–(34), we obtain the optimum value of $P_{k,o}$, which are reported in Table 7.

As can be seen in Table 7, if there is a restriction on VaR, the maximum profit will decrease by 25%. The total lost profit under VaR restriction is 5728251.77, which is calculated based on Tables 5 and 7. There is a trade-off between maximum expected profit and this restriction. As the VaR is limited more, the expected value is decreased more since the price of products is decreased. Therefore, high expected profit comes with high value at risks. So far, it has been assumed that we have only one decision variable whereas other variables are considered to be known. To study the effect of these variables on the optimal online price, we can draw Figs. 2–13. Fig. 2, Fig. 4, and Fig. 6 show relationship between profit, variance, and VaR with warranty length for product model 1001. Fig. 3, Fig. 5, and Fig. 7 show relationship between profit, variance, and VaR with warranty usage for product model 1001. Fig. 8, Fig. 9, and Fig. 10 show relationship between total profit and warranty length and usage. Also, Fig. 11, Fig. 12, and Fig. 13 show relationship between total profit and offline price and warranty length. The behavior of these functions for other models is similar. In the next section, we can see that paying attention to the dependencies between products leads to better pricing and thus more profit.

Fig. 8 indicates that an increase in warranty length and usage could positively and negatively effect on total profit. Indeed, if we increase the warranty length and usage until they reach their optimal point, the total profit increases, while after that point, the total profit decreases. Similarly, such a relationship exists between total profit and offline price and warranty length (Fig. 11). Fig. 9 indicates that an increase in warranty length and usage leads to an increase in variance. An increase in warranty length and usage attracts more customers and simultaneously increase the number of warranty claims. That is why such a relationship exists.

Table 7
Optimum values of $P^2_{k,o}$ in the VaR model by considering a restriction on ζ.

Model (k)	ζ	$P^2_{k,o}$	$E[W_k]$	$VaR_{0.05}(W_k)$
1001	3,000,000	3286.8	3012914.209	3,000,000
1002	3,000,000	2237.9	3008038.381	3,000,000
1003	3,000,000	1520.8	2827900.000	3,000,000
1004	3,000,000	3718.5	1871000.000	1,703,008
1005	3,000,000	4971.7	3008499.303	3,000,000

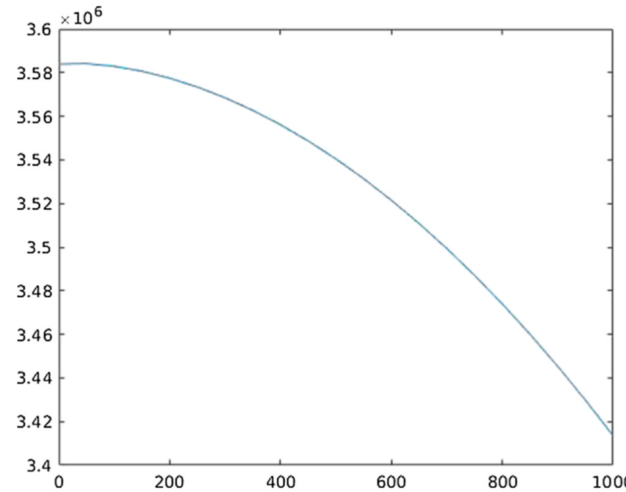

Fig. 2. Relationship between total profit and warranty length in the optimal online price of product model 1001.

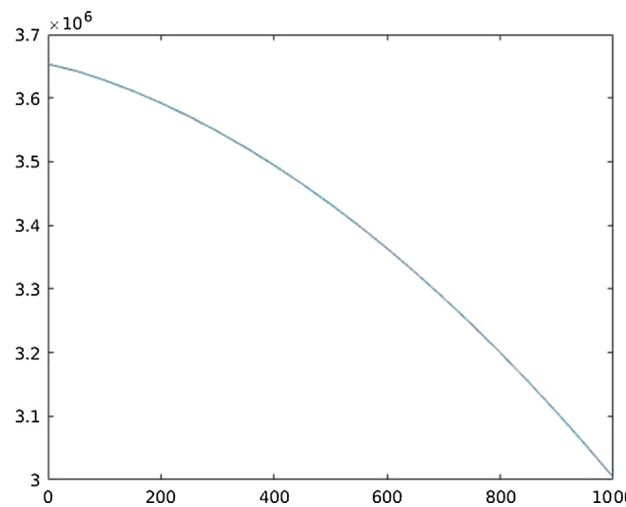

Fig. 3. Relationship between total profit and warranty usage in the optimal online price of product model 1001.

Fig. 12 also indicates such a relationship between variance and offline price and warranty length. Fig. 10 reveals a linear relationship between VaR and warranty length and usage, while Fig. 13 indicates a quadratic relationship between VaR and offline price and warranty length, which is because of the quadratic relationship between VaR and

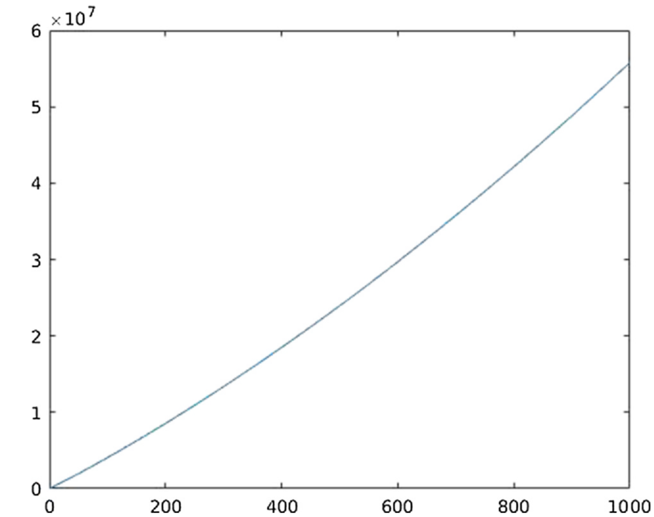

Fig. 4. Relationship between variance and warranty length in the optimal online price of product model 1001.

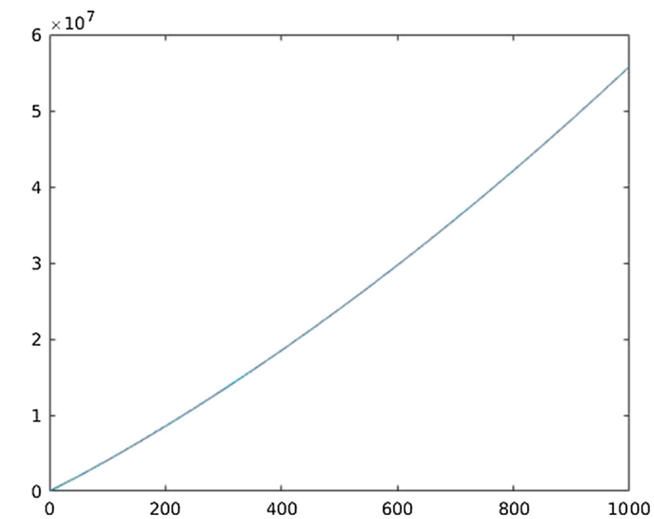

Fig. 5. Relationship between variance and warranty usage in the optimal online price of product model 1001.

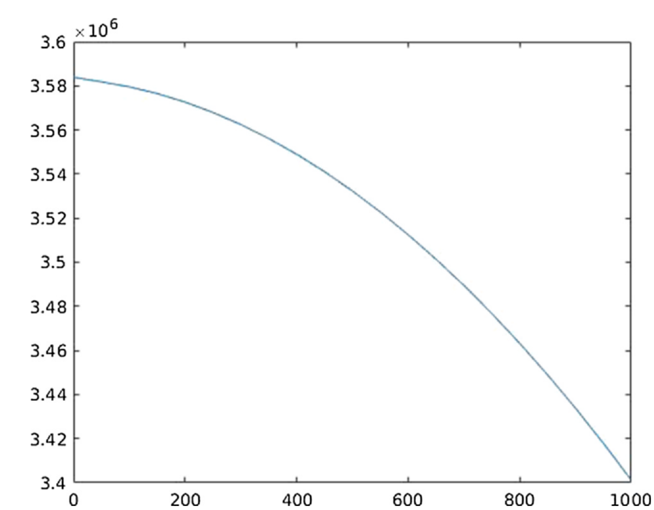

Fig. 6. Relationship between VaR and warranty length in the optimal online price of product model 1001.

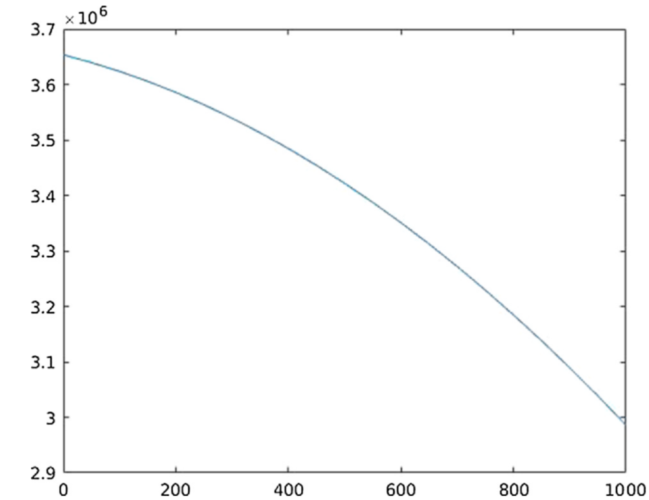

Fig. 7. Relationship between VaR and warranty usage in the optimal online price of product model 1001.

offline price. An increase in the offline price leads to an increase in the amount of soled products until we reach optimal offline price, which leads to an increase in profit and VaR. After that, the amount of soled products decreases, which leads to a decrease in profit and VaR.

Also, we can determine the offline price by considering constant values for other variables. The results are shown in Table 8 and Table 9. For this, 2800, 1800, 1300, 3000, and 4500 are considered for $P_{k,o}$ of model 1001, 1002, 1003, 1004, and 1005 respectively. The optimum value of warranty length and warranty usage can be obtained using similar calculations.

5.2. Multi products case

As mentioned before, there are dependencies between different products. For considering these dependencies, in Sections 3.3 and 4.2 formulas are developed. In this section, the modeling's results are applied to this instance to improve the pricing decision.

As it was mentioned before, copula is a useful tool for modeling the dependence of products. In this study, due to the potential upper tail-dependence and non-elliptical marginal distributions, Gumbel copula is implemented. For more information about copulas, one can see (Hutchinson, 1990). The JD of NFC is expressed as follows:

$$C(z_1, z_1, \ldots, z_n; \theta) = \exp\left\{-[(-\ln u_1)^{\theta} + (-\ln u_2)^{\theta} + (-\ln u_3)^{\theta} + (-\ln u_4)^{\theta} + (-\ln u_5)^{\theta}]^{\frac{1}{\theta}}\right\} \tag{56}$$

where $u_i = F^{(N_i)}(z_i)$ and $1 \leqslant \theta \leqslant +\infty$ is copula parameter which denotes the correlation between products. Also, $\theta = 1$ means that there is no correlation between products. The density of the JD is presented by Eq. (40). The CDF of the total NFCs of all products $N = \sum_{i=1}^{5} N_i$ is presented by Eq. (57):

$$F_N(z) = \sum c(z_1, z_1, \ldots, z_n) \tag{57}$$

Suppose that the online prices of products are set (in this case we suppose the prices are the prices of Table 7), Then the VaR of this scenario can be determined using Eqs. (40), (56), and (57), and the results are presented in Table 10. Also, an approximation relationship between θ and the VaR is shown in Fig. 14.

As can be seen in Table 10, taking into account the dependency between products leads to a realization of the VaR. This has two advantages: first, if the company wants to enter the market with a certain VaR (for example13703008), it can increase its prices to reach the desired VaR. Second, if the company wants to enter the market with the

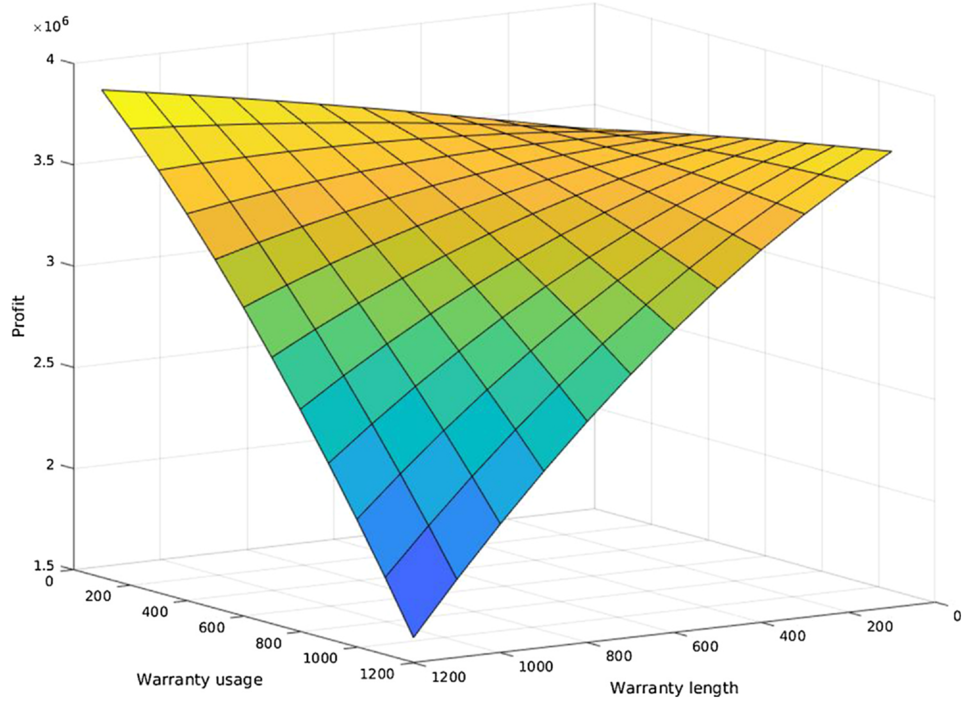

Fig. 8. Relationship between total profit and warranty length and usage in the optimal online price of model 1001.

optimal prices for the state of the products without interdependence, it can enter the market more confidently.

6. Conclusions

Manufacturers often offer different kinds of products or different models of a product through various sales channels like online and offline channels. Regularities or competitions force manufacturers to provide services such as warranties for their products which cause additional costs. This study is conducted to investigate the effect of warranty claim rates of products with similar components on the optimal pricing. Sales are made through online and offline channels and a two-dimensional warranty policy including warranty age and warranty usage of products are considered. The proposed model seeks to maximize company profits under conditions in which the VaR does not exceed by a certain value. Two models for situations where the relationship between product warranty claims is considered and not considered are developed and numerical experiment showed that considering the covariance between products leads to better pricing and more profit. Also, due to the high complexity of covariance, Copulas has been used to estimate covariance and value at risk. In this study, the warranty claims and warranty costs are assumed that follow the NHPP

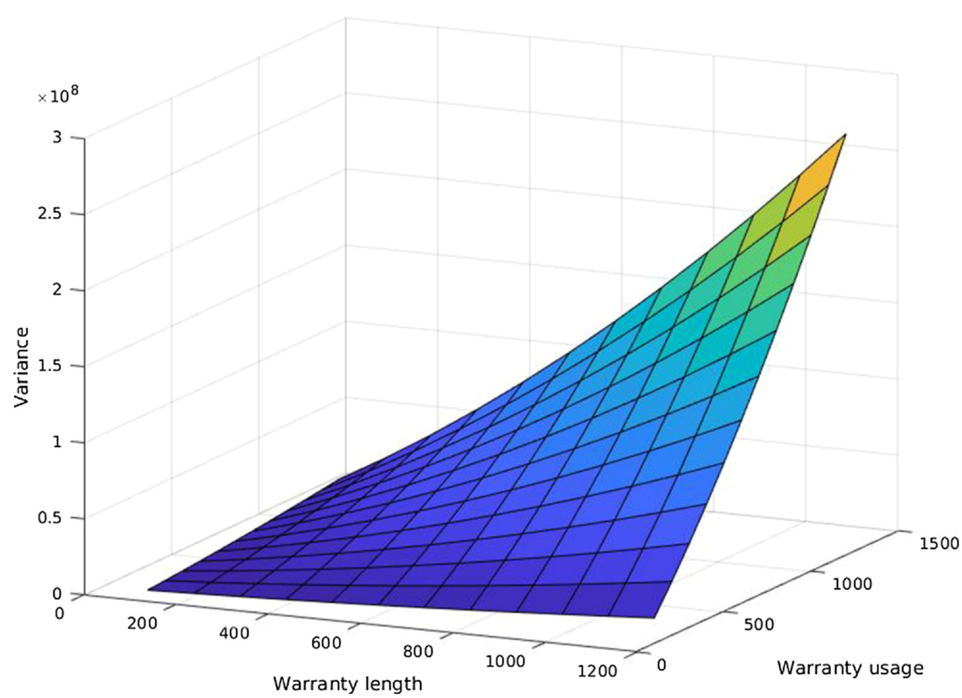

Fig. 9. Relationship between variance and warranty length and usage in the optimal online price of model 1001.

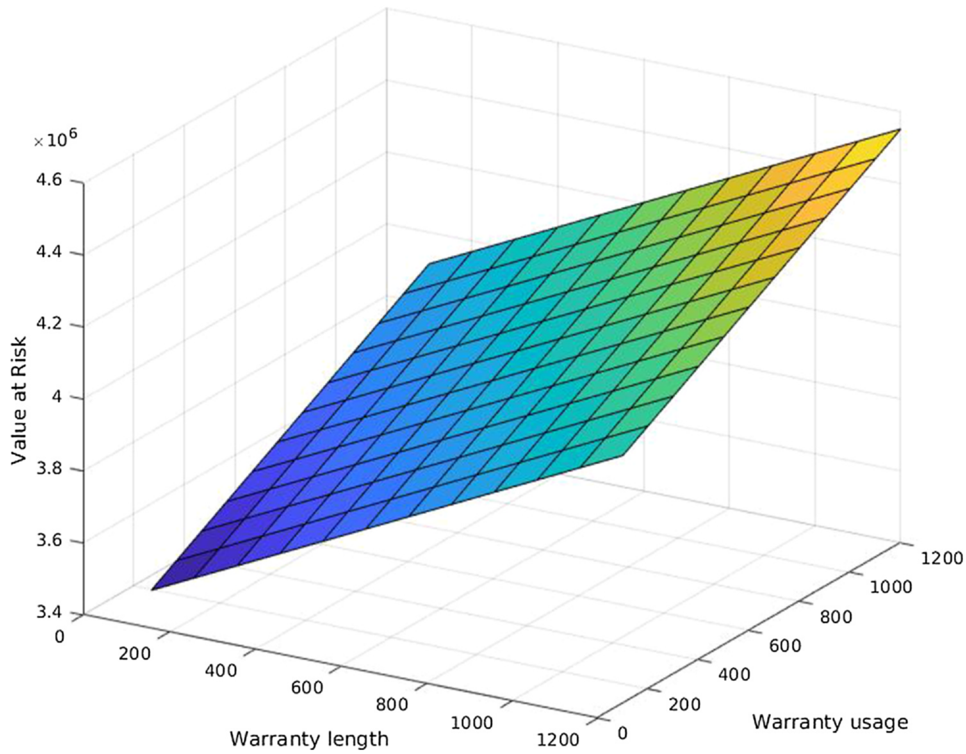

Fig. 10. Relationship between VaR and warranty length and usage in the optimal online price of model 1001.

and log-normal distribution, which may not model some industries, particularly where the warranty is not rectified by minimal repair. Therefore, it is suggested to investigate the problem under such assumptions. Moreover, one can apply CVaR and compare the result to VaR. It is also proposed to explore various Coppola models according to their features and problem features. Also, one can analyze the effect of pricing and warranty policy on the manufacturer's sale behavior, particularly on the preference of manufacturer's sales amount through online or offline channels respect to a fixed quantity of products on hand.

CRediT authorship contribution statement

Ata Allah Taleizadeh: Conceptualization, Methodology, Writing - review & editing, Supervision. **Mahdi Mokhtarzadeh:** Data curation, Writing.

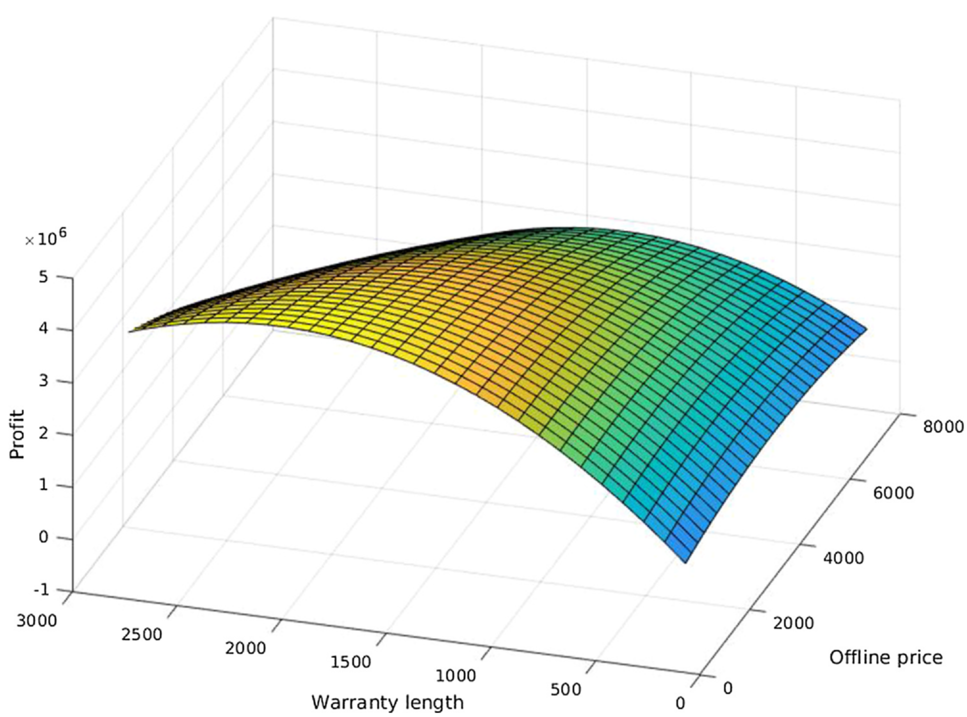

Fig. 11. Relationship between total profit and offline price and warranty length in the optimal online price of model 1001.

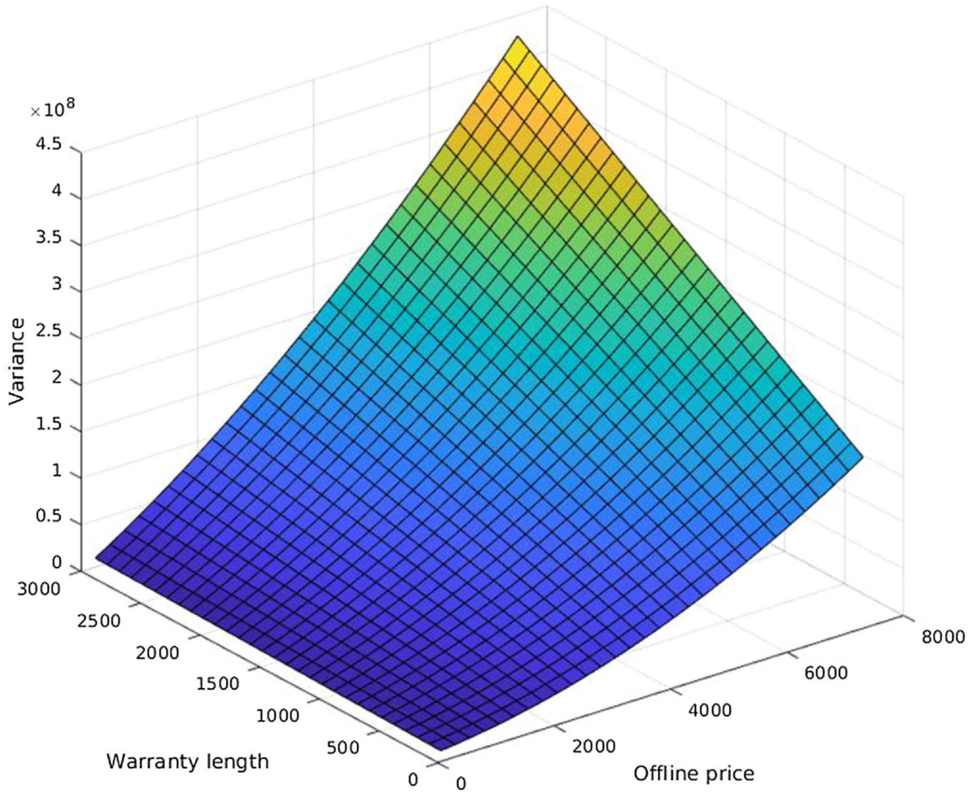

Fig. 12. Relationship between variance and offline price and warranty length in the optimal online price of model 1001.

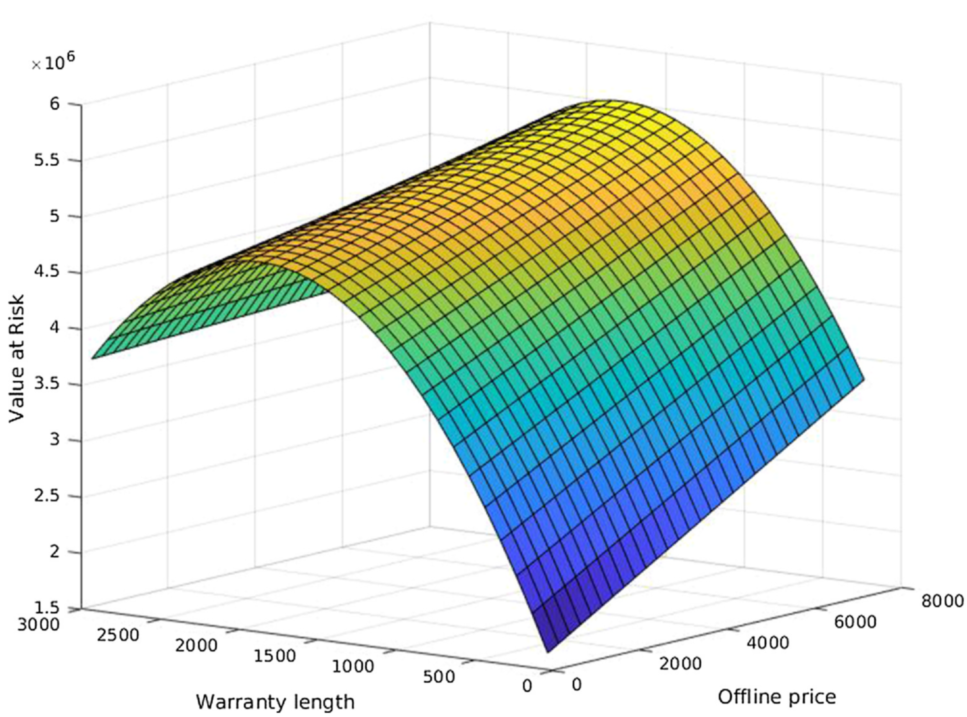

Fig. 13. Relationship between VaR and offline price and warranty length in the optimal online price of model 1001.

Table 8
Optimum values of $P_{k,f}$ in the VaR model without considering a restriction on ζ.

Model (k)	$P_{k,f}$	$E[W_k]$	$VaR_{0.05}(W_k)$
1001	3151	2.6940e+06	4,161,249
1002	5431	3.5773e+06	4,024,698
1003	3554	2.5443e+06	3,372,437
1004	3059	1.8546e+06	2,223,151
1005	2623	2.8375e+06	2,800,200

Table 9
Optimum values of $P_{k,f}$ in the VaR model by considering a restriction on ζ.

Model (k)	ζ	$P_{k,f}$	$E[W_k]$	$VaR_{0.05}(W_k)$
1001	3,000,000	1644	2.3531e+06	3,000,000
1002	3,000,000	2388	2.6510e+06	3,000,000
1003	3,000,000	4680	2.3542e+06	3,000,000
1004	3,000,000	3059	1.8546e+06	2,223,151
1005	3,000,000	2623	2.8375e+06	2,800,200

Appendix A. The proof of Proposition 1

Proving. Assume that there is more than one decision variable. Consider $P_{k,o}$ and T_k as decision variables. Then, the Hessian matrix (HM) of $E[W_k]$is as follows:

$$\begin{bmatrix} -2\beta_{k,o} & \lambda_k + \left(\frac{\beta_{k,o}\partial_{k,1}\mu_k}{\theta_{k,1}}(\frac{T_k}{\theta_{k,1}})^{\partial_{k,1}-1}(\frac{U_k}{\theta_{k,2}})^{\partial_{k,1}}\right) \\ \lambda_k + \left(\frac{\beta_{k,o}\partial_{k,1}\mu_k}{\theta_{k,1}}(\frac{T_k}{\theta_{k,1}})^{\partial_{k,1}-1}(\frac{U_k}{\theta_{k,2}})^{\partial_{k,1}}\right) & -\left(\frac{2\lambda_k\partial_{k,1}\mu_k}{\theta_{k,1}}(\frac{T_k}{\theta_{k,1}})^{\partial_{k,1}-1}(\frac{U_k}{\theta_{k,2}})^{\partial_{k,1}}\right) - \left(\frac{\partial_{k,1}\mu_k}{\theta_{k,1}^2}(\frac{T_k}{\theta_{k,1}})^{\partial_{k,1}-2}(\frac{U_k}{\theta_{k,2}})^{\partial_{k,1}}(C_{k,o} - \beta_{k,o}P_{k,o} + \gamma_k U_k + \lambda_k T_k + \alpha_f\beta_{k,f}P_{k,f})\right) \end{bmatrix} \quad (1.1)$$

Then by calculating the eigenvalues of $E[W_k]$, it can be seen that $x_1 < 0$ and $x_2 > 0$. The global minimum of $E[W_k]$ does not exist because the HM is indefinite. These calculations are available for interested readers upon request.

Also, the HM of $Var(W_k)$is as follows:

$$\begin{bmatrix} 0 & -\left(\frac{\beta_{k,o}\partial_{k,1}}{\theta_{k,1}}(\frac{T_k}{\theta_{k,1}})^{\partial_{k,1}-1}(\frac{U_k}{\theta_{k,2}})^{\partial_{k,1}}\right)(\sigma_k^2 + \mu_k^2) \\ -\left(\frac{\beta_{k,o}\partial_{k,1}}{\theta_{k,1}}(\frac{T_k}{\theta_{k,1}})^{\partial_{k,1}-1}(\frac{U_k}{\theta_{k,2}})^{\partial_{k,1}}\right)(\sigma_k^2 + \mu_k^2) & -\left(\frac{2\lambda_k\partial_{k,1}}{\theta_{k,1}}(\frac{T_k}{\theta_{k,1}})^{\partial_{k,1}-1}(\frac{U_k}{\theta_{k,2}})^{\partial_{k,1}}(\sigma_k^2 + \mu_k^2)\right) \\ & -\left(\frac{\partial_{k,1}}{\theta_{k,1}^2}(\frac{T_k}{\theta_{k,1}})^{\partial_{k,1}-2}(\frac{U_k}{\theta_{k,2}})^{\partial_{k,1}}(\sigma_k^2 + \mu_k^2)(C_{k,o} - \beta_{k,o}P_{k,o} + \gamma_k U_k + \lambda_k T_k + \alpha_f\beta_{k,f}P_{k,f})\right) \end{bmatrix} \quad (1.2)$$

Then by calculating the eigenvalues of $Var(W_k)$, it can be seen that $x_1 < 0$ and $x_2 > 0$. The HM is indefinite and the feasible region of $(P_{k,o} + T_k)$ defined by the constraint is therefore infinite. As a result, the objective function's global maximum does not exist. It can be shown for other states of this problem like this one that if more than one decision variable exists then the global maximum does not exist.

Appendix B. Proposition 2's proving

Proving. From Appendix A, we know that just one variable can act as the decision variable. Assume that $P_{k,o}$ is decision variable and others are known, then the first-order derivative (FOD) of the objective function $E[W_k](P_{k,o})$ is as follows:

$$\frac{\partial E[W_k](P_{k,o})}{\partial P_{k,o}} = C_{k,o} - \beta_{k,o}\left(P_{k,o} - \mu_k(\frac{T_k}{\theta_{k,1}})^{\partial_{k,1}}(\frac{U_k}{\theta_{k,2}})^{\partial_{k,2}}\right) + \gamma_k U_k + \lambda_k T_k + \alpha_f\beta_{k,f}P_{k,f} + \alpha_o\beta_{k,o}P_{k,f} \quad (2.1)$$

The second-order derivative (SOD) of the $E[W_k](P_{k,o})$is as follows:

$$\frac{\partial^2 E[W_k](P_{k,o})}{\partial P_{k,o}^2} = -2\beta_{k,o} < 0 \quad (2.2)$$

It implies that $E[W_k](P_{k,o})$is concave for $P_{k,o} \geqslant 0$which means $E[W_k]$achieves the global maximum at $P_{k,o} = \frac{C_{k,o} + \lambda_k T_k + \gamma_k U_k + \alpha_f\beta_{k,f}P_{k,f} - \beta_{k,o}(\frac{T_k}{\theta_{k,1}})^{\partial_{k,1}}(\frac{U_k}{\theta_{k,2}})^{\partial_{k,2}}\mu_k}{2\beta_{k,o} - \alpha_o\beta_{k,o}P_{k,f}}$. Also, if $P_{k,f}$is known, this procedure can apply to find the maximum value of $P_{k,f}$.

Table 10
The VaR of total profit in multi products mode under differentθ.

Copula parameter (θ)	1	2	3	4	5	6
VaR	13,703,008	13,603,008	13,574,358	13,563,045	13,559,905	13,557,058

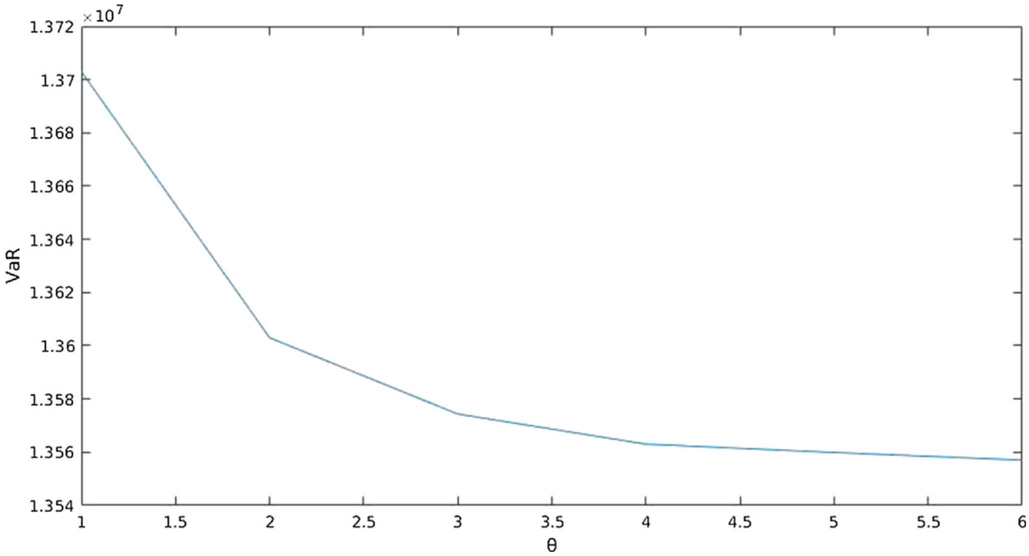

Fig 14. Total VaR against θ, in case of determined $P_{k,o}$.

Now, Assume that T_k is decision variable and others are known, then the FOD of $E[W_k](T_k)$ is as follows:

$$\frac{\partial E[W_k](T_k)}{\partial T_k} = \lambda_k\left(P_{k,o} - \mu_k(\frac{T_k}{\theta_{k,1}})^{\partial_{k,1}}(\frac{U_k}{\theta_{k,2}})^{\partial_{k,2}}\right) - \alpha_o\lambda_k P_{k,f} - \frac{\left(\partial_{k,1}\mu_k(\frac{T_k}{\theta_{k,1}})^{\partial_{k,1}-1}(\frac{U_k}{\theta_{k,2}})^{\partial_{k,2}}(C_{k,o} - \beta_{k,o}P_{k,o} + \gamma_k U_k + \lambda_k T_k + \alpha_f\beta_{k,f}P_{k,f})\right)}{\partial_{k,1}} \tag{2.3}$$

The SOD of $E[W_k](T_k)$is as follows:

$$\frac{\partial^2 E[W_k](T_k)}{\partial T_k^2} = \frac{-\left(2\frac{\partial_{k,1}\mu_k(\frac{T_k}{\theta_{k,1}})^{\partial_{k,1}-1}(\frac{U_k}{\theta_{k,2}})^{\partial_{k,2}}}{\theta_{k,1}} - \partial_{k,1}(\partial_{k,1}-1)\mu_k(\frac{T_k}{\theta_{k,1}})^{\partial_{k,1}-2}(\frac{U_k}{\theta_{k,2}})^{\partial_{k,2}}(C_{k,o} - \beta_{k,o}P_{k,o} + \gamma_k U_k + \lambda_k T_k + \alpha_f\beta_{k,f}P_{k,f})\right)}{\partial_{k,1}^2} < 0 \tag{2.4}$$

It implies that $E[W_k](T_k)$is concave for $T_k \geqslant 0$. Also, the FOD of $Var(W_k)(T_k)$is as follows:

$$\frac{\partial Var(W_k)(T_k)}{\partial T_k} = \lambda_k(\frac{T_k}{\theta_{k,1}})^{\partial_{k,1}}(\frac{U_k}{\theta_{k,2}})^{\partial_{k,2}}(\sigma_k^2 + \mu_k^2) + \partial_{k,1}(\frac{T_k}{\theta_{k,1}})^{\partial_{k,1}-1}(\frac{U_k}{\theta_{k,2}})^{\partial_{k,2}}(\sigma_k^2 + \mu_k^2)(C_{k,o} - \beta_{k,o}P_{k,o} + \gamma_k U_k + \lambda_k T_k + \alpha_f\beta_{k,f}P_{k,f})/\theta_{k,1} > 0 \tag{2.5}$$

The SOD of the $Var(W_k)(T_k)$is as follows:

$$\frac{\partial^2 Var(W_k)(T_k)}{\partial T_k^2} = \frac{2\partial_{k,1}\lambda_k\left(\frac{T_k}{\theta_{k,1}}\right)^{\partial_{k,1}-1}\left(\frac{U_k}{\theta_{k,2}}\right)^{\partial_{k,2}}(\sigma_k^2 + \mu_k^2)}{\theta_{k,1}} - \partial_{k,1}(\partial_{k,1} - 1)\left(\frac{T_k}{\theta_{k,1}}\right)^{\partial_{k,1}-2}\left(\frac{U_k}{\theta_{k,2}}\right)^{\partial_{k,2}}(\sigma_k^2 + \mu_k^2)(C_{k,o} - \beta_{k,o}P_{k,o} + \gamma_k U_k + \lambda_k T_k + \alpha_f\beta_{k,f}P_{k,f})/\theta_{k,1} > 0 \tag{2.6}$$

It means that $Var(W_k)(T_k)$is convex and monotonously grows for $T_k \geqslant 0$and the feasible space of $T_k$restricted by $VaR_\theta(W_k) \leqslant \zeta$ is infinite. So, there is an optimal solution to this problem. Also, if $U_k$is not known, this procedure can apply to find the maximum value of U_k.

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