This article has been accepted for inclusion in a future issue of this journal. Content is final as presented, with the exception of pagination.

Reliability Importance Measures for Network Based on Failure Counting Process

Yongjun Du, Shubin Si, *Senior Member, IEEE*, and Tongdan Jin, *Member, IEEE*

***Abstract*—Traditional importance measures seldom consider how the number of failed components influences the network reliability. This paper proposes two importance measures under the circumstance that the failure sequence of the components follows a counting process. The first importance measure aims to assess the contribution of the individual component (edge) to the network failure. The second evaluates the contribution of the individual component to the network functionality. Both importance measures are time-dependent functions, and their values are jointly determined by the network structure and the distribution of the number of failed components at a particular time. We prove that the proposed importance measures are able to generate consistent rankings based on edge's impact on the network reliability behavior. When networks possess special structure or the number of failed edges follows the special distribution, the rankings are coincident with the results generated from some traditional importance measures. When component's failure sequence follows a saturated nonhomogeneous Poisson process, the proposed importance measures are equivalent to the structural importance measure as time approaches zero or infinite. Finally, numerical examples are provided to demonstrate the application and performance of the proposed measures.**

***Index Terms*—C-spectrum, D-spectrum, dynamic importance measure (IM), network reliability, saturated nonhomogeneous Poisson process (SNHPP).**

ABBREVIATIONS

IM	Importance measure.
NHPP	Nonhomogeneous Poisson process.
SNHPP	Saturated nonhomogeneous Poisson process.

NOTATION

N	Graph representing a network.
n	Number of edges in a network.
i	Index for edge *i* for $i = 1, 2, \ldots, n$.
T	Lifetime of the network.
K	Terminal set of the network.
K-connected	If all terminals are connected to each other, the network is called K-connected.
cut	Set of edges whose removal disconnects the terminals in K.
path	Set of consecutive edges connecting all the terminals in *K*.
$\lambda(t)$	Intensity function in the NHPP.
$\Lambda(t)$	Mean value function in the NHPP.
$N(t)$	Number of failed edges at time *t*.
$C_i(P_i)$	Set of cuts (paths) that include edge *i*.
$C(k)(P(k))$	Set of cuts (paths) of size *k*.
$C_i(k)(P_i(k))$	Set of cuts (paths) of size *k* that includes edge *i*.
$C_{(i)}(k)(P_{(i)}(k))$	Set of cuts (paths) of size *k* that does not include edge *i*.
$F(k)(F'(k))$	D-spectrum (C-spectrum) of the network, it is the probability that the network is down (up) if exactly *k* randomly chosen edges are down (up).
$F(k, 0_i)$	D-spectrum of edge *i*, it is the probability that the network is down if exactly *k* randomly chosen edges, including edge *i*, are down. Note that $F(1, 0_i) \leq F(2, 0_i) \leq F(3, 0_i) \leq \ldots \leq F(n, 0_i)$.
$F(k, 1_i)$	$= F(k) - F(k, 0_i)$.
$F'(k, 1_i)$	C-spectrum of edge *i*, it is the probability that the network is up if exactly *k* randomly chosen edges, including edge *i*, are up. Note that $F'(1, 1_i) \leq F'(2, 1_i) \leq F'(3, 1_i) \leq \ldots \leq F'(n, 1_i)$.
$F'(k, 0_i)$	$= F'(k) - F'(k, 1_i)$.
$k_{\min\{i,j\}}(k'_{\min\{i,j\}})$	$= \min\{k : F(k, 0_i) \neq F(k, 0_j)\}$ $(= \min\{k : F'(n-k, 1_i) \neq F'(n-k, 1_j)\})$.
$k_{\max\{i,j\}}(k'_{\max\{i,j\}})$	$= \max\{k : F(k, 0_i) \neq F(k, 0_j)\}$ $(= \max\{k : F'(n-k, 1_i) \neq F'(n-k, 1_j)\})$.
$k_i(k'_i)$	$= \min\{k : F(k, 0_i) \neq 0\}(= \min\{k : F'(k, 1_i) \neq 0\})$.
$a(a')$	$= \min\{k : F(k) \neq 0\}(= \min\{k : F'(k) \neq 0\})$.
$I^c(i,t)(I^p(i,t))$	*C*-importance measure (*p*-importance measure) of edge *i* at time *t*.

Manuscript received September 20, 2017; revised March 9, 2018; accepted August 2, 2018. This work was supported in part by the National Natural Science Foundation of China under Grant 71771186 and Grant 71631001 and in part by the 111 Project under Grant B13044. Associate Editor: H. Waeselynck. *(Corresponding author: Shubin Si.)*

Y. Du and S. Si are with the School of Mechanical Engineering, Northwestern Polytechnical University, Xi'an 710072, China (e-mail: alphadu@yeah.net; sisb@nwpu.edu.cn).

T. Jin is with the Ingram School of Engineering, Texas State University, San Marcos, TX 78666 USA (e-mail: tj17@txstate.edu).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TR.2018.2864563

I. Introduction

Since the inception of Birnbaum reliability importance measure (IM) in 1969 [1], IMs have become an effective method to identify the weakness of a system. They are also used to guide the optimal reliability and redundancy allocation for complex system design. Based on Birnbaum's method, different variations have been developed thereafter, including Bayesian IM [2], criticality IM [3], redundancy IM [4], [5], Fussell–Vesely (FV) IM [6], [7], Barlow–Proschan IM [8], and joint IM [9]–[12]. Based on the values of the IM, components can be ranked with respect to the influence they have on the system reliability. Furthermore, these measures can serve as a useful tool to identify component criticality with respect to the system reliability and operational safety.

A network can be treated as a graph comprising multiple nodes and edges. Some particular nodes are called terminals. Nodes can represent telecommunications switches, computer servers, railway stations, and power plants. Edges can represent telecommunication links, road, railways, and power lines. This paper assumes that nodes are 100% reliable, while the edges are prone to failures. An edge failure means that the edge is disconnected between two adjacent nodes. The reliability of network is defined as the probability that all terminals are connected among each other.

IMs for network systems can be built upon the cut or path theory. Based on [13], *c*-type IMs are those that can be defined by cuts, while *p*-type IMs are those that can be defined by paths. For example, two types of FV IMs have been developed based on cut and path sets. Given that the network is down, the *c*-type FV IM [6], [7] is defined as the probability that the failure of an edge contributes to network failure. Given that the network is up, the *p*-type FV IM [14] calculates the probability that at least one minimal path including a fixed edge is operational. Both the *c*-type and the *p*-type FV IMs are dependent on the network structure and edge reliability. Similarly, permutation IM [15], domination IM [16], and H-IM [17] can be classified into *c*-type or *p*-type as well. In general, a cut IM [18] belongs to *c*-type; a path IM [13] belongs to *p*-type.

IMs for networks are also proposed to calculate the reliability importance of edges. For instance, Page and Perry [19] developed a contract-delete IM and a link IM for network reliability analysis. The former is established on the size and number of minimum cut (path) including the target edge. The latter considers the contribution of an edge to the network reliability. Hong and Lie [20] defined a joint IM that assesses the interactions of two edges and their contribution to the network reliability. Gertsbakh and Shpungin [21] extended the traditional Birnbaum IM to network structure for exploring the edge importance.

One important concept in computing network reliability and IM is the network spectrum that depends solely on the network structure [22], [23]. Two types of network spectra are often used: construction-spectrum (C-spectrum) and destruction-spectrum (D-spectrum). The concept of spectra provides a new way to compute the Birnbaum IM and the joint reliability IM [21]. Since the analytical solutions are intractable, Monte Carlo simulation or approximation techniques are often used to estimate the spectra.

For most networks in real applications, the number of edge failures at a particular time is a random variable and can be treated as a stochastic counting process. Although various IMs have been proposed, IMs based on counting processes are rarely reported in the literature. That is, existing IMs are inadequate to address how the network reliability evolves dynamically when edges fail sequentially according to a counting process. This paper proposes two time-dependent IMs to capture the dynamic nature of network reliability. The first IM is built upon the cut and is referred to as the *c*-importance measure or simply *c*-IM. It evaluates the contribution of an edge to the network failure at time t and is represented in the form of D-spectrum. The second IM is built upon the path and is called *p*-importance measure or simply *p*-IM. It quantifies the contribution of an edge to the network functionality at time t and is represented in the form of C-spectrum.

Recently, new attention has been paid to using a stochastic process of component failures to model the network reliability. Gertsbakh and Shpungin [24] proposed a probabilistic model in which failures of components follow a renewal process. Based on their model, Zarezadeh and Asadi [25] investigated various properties of network reliability under different failure scenarios. In [26], the stochastic properties of network reliability were investigated under random shocks that arrive in accordance with a counting process. In these approaches, probabilistic information such as edge reliability or lifetime distribution is not required. This is a great advantage because probabilistic information of individual edges sometimes is not easy to obtain if the sample size or testing time is limited. In this paper, the proposed IMs generate the edge ranking based on their contribution to the network failure or functionality. These IMs depend on the structure of the network and the distribution of the number of failed edges at time t, instead of the probabilistic information of individual edge. Moreover, we have proved that the proposed IMs generate the same rankings, which are coincident to certain classic IMs under some conditions. Finally, if edge failures follow an SNHPP, the proposed IMs rankings become the structural ranking, i.e., depending only on the network structure as time t approaches zero or infinite. The assumptions made in the proposed methods are summarized as follows.

1) The edges of network are subject to failure, while the nodes are always reliable. Hence, the term "components" are used to refer to the edges of network.
2) This paper deals with binary network. That is, the network and edges are either down or up.
3) The failures of the edges are independent and identically distributed.
4) Failures of edges arrive according to a counting process.

The remainder of the paper is structured as follows. In Section II, some basic concepts and two new IMs are introduced. Moreover, the relationships among some IMs are discussed. Section III focuses on a failure process in which the numbers of failed edges follow an SNHPP. Section IV presents two examples to demonstrate how the newly proposed IMs can effectively assist in obtaining the criticality of edges regarding the reliability of the network. Finally, conclusions are provided in Section V. All proofs can be found in the Appendix.

II. Importance Measures Under a Failure Counting Process

Traditional network IMs depend on the structure of the network or the probabilistic information of individual edge. However, the probabilistic information is not easy to obtain, especially for small component sample or limited testing time. In this section, we propose two new IMs for the network in which failures of edges occur according to a counting process $\{N(t), t \geq 0\}$, where $N(t)$ denotes the number of failed edges in interval $[0, t]$.

A. Basic Notions and Definitions

Network N is a triplet $N = (V, E, K)$, where V is a set of nodes, E is a set of edges with $|E| = n$, and K is a set of special nodes called terminals, $K \subseteq V$. Assume that all nodes are failure-free, whereas all edges are prone to stochastic failure. When an edge fails, it is disconnected from two adjacent nodes. If there is at least one path connecting any pair of nodes in K, the network is called K-connected. This paper deals with K-connected terminal reliability criterion. That is, the network is defined as up state if the network is K-connected. Otherwise, it is in down state. For example, $K = \{s, t\}$ means the network only has two terminals s and t. The network is up if and only if there is a set of operational edges connecting s and t. If $K = V$, the network is up if and only if there is a set of operational edges connecting any pair nodes of V. The network reliability R is the probability that the network is in the up state.

Many IMs are proposed based on the cut theory. A cut is a subset of edges whose removal disconnects at least one pair of terminals in K. A cut is down means that all edges belonging to the cut are not operational. With respect to the terminal set K, let C_i denote the set of cuts including edge i, and $C(d)$ denote the set of cuts of cardinality d. In addition, $C_i(d)$ and $C_{(i)}(d)$ are, respectively, the set of cuts of size d that include and that do not include edge i. Moreover, a path is a subset of edges that ensure K-connectedness of the network. A path is up means that all edges belonging to the path are functional. Similarly, $P_i, P(d), P_i(d)$, and $P_{(i)}(d)$ can be defined according to the definition of $C_i, C(d), C_i(d)$, and $C_{(i)}(d)$.

Spectra play a critical role in the assessment of network reliability and IMs. The value of a spectrum depends only on the network structure. Hence, they are called structural invariants. The definitions of D-spectrum and C-spectrum are given as follows.

Definition 1: Given a network comprises n edges. The collection of $\{F(k) = |C(k)|\binom{n}{k}^{-1}\}, \quad k = 1, \ldots, n$ is called the destruction spectrum (D-spectrum). Furthermore, the collection of $\{F(k, 0_i) = |C_i(k)|\binom{n}{k}^{-1}\}, \quad i = 1, \ldots, n, k = 1, \ldots, n$ is called the D-spectrum of edge i.

According to [23], $F(k)$ can be interpreted as the probability that the network is down if exactly k randomly chosen edges are down. Since $|C(k)| = |C_i(k)| + |C_{(i)}(k)|$, the probability $F(k)$ can be written as $F(k) = F(k, 0_i) + F(k, 1_i)$, where $F(k, 0_i)$ $(F(k, 1_i))$ is the probability that the network is down if exactly k randomly chosen edges are down, and edges i is down (up).

Definition 2: Given a network comprises n edges. The collection of $\{F'(k) = |P(k)|\binom{n}{k}^{-1}\}, \quad k = 1, \ldots, n$ is called the construction spectrum (C-spectrum). Furthermore, the collection of $\{F'(k, 1_i) = |P_i(k)|\binom{n}{k}^{-1}\}, \quad i = 1, \ldots, n, k = 1, \ldots, n$ is called the C-spectrum of edge i.

Similarly, $F'(k)$ represents the probability that the network is up if exactly k randomly chosen edges are up [23]. By realizing that $|P(k)| = |P_i(k)| + |P_{(i)}(k)|$, the probability $F'(k)$ can be written as $F'(k) = F'(k, 0_i) + F'(k, 1_i)$, where $F'(k, 0_i)$ $(F'(k, 1_i))$ is the probability that the network is up if exactly k randomly chosen edges are up, and edges i is down (up).

Suppose T is the lifetime of the network, and the number of failed edges increases according to a counting process $\{N(t), t \geq 0\}$. By combining the law of total probability with the definition of $F(k)$, the probability that the network is down at time t can be expressed as

$$P(T \leq t) = \sum_{k=0}^{n} P(N(t) = k)F(k). \tag{1}$$

Furthermore

$$P(\text{a cut } C \in C_i \text{ is down at } t) = \sum_{k=0}^{n} P(N(t) = k) \times F(k, 0_i) \tag{2}$$

where a cut $C \in C_i$ is down means that all edges in C have failed.

Similarly, by combining the law of total probability with the definition of C-spectrum, the probability of the network is up at time t can be represented as

$$P(T > t) = \sum_{k=0}^{n} P(N(t) = k)F'(n - k). \tag{3}$$

Furthermore

$$P(\text{a path } P \in P_i \text{ is up at } t) = \sum_{k=0}^{n} P(N(t) = k)F'(n - k, 1_i) \tag{4}$$

where a path $P \in P_i$ is up means that all edges in P are good or functional.

When a network fails, it is interesting to address two questions. First, how to identify the cut actually that brings the network in the down state. Second, what is the contribution of edge i to the down state of the network. To answer both questions, we propose the following definition.

Definition 3: Consider a network with n edges. Suppose the occurrence of edge failures follow a counting process $\{N(t), t \geq 0\}$. The c-IM of edge i, denoted by $I^c(i, t)$, is the conditional probability that a cut including edge i is down at time t, given

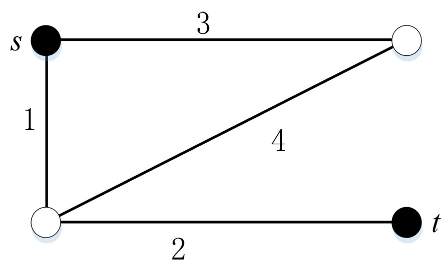

Fig. 1. Simple network.

that the network is down at time *t*. Mathematically, we have

$$\begin{aligned} I^c(i,t) &= P(\text{a cut } C \in C_i \text{ is down at } t \,|\text{the network is down at } t) \\ &= \frac{P(\text{a cut } C \in C_i \text{ is down at } t)}{P(\text{the network down at } t)} \\ &= \frac{\sum_{k=0}^{n} P(N(t)=k)F(k,0_i)}{\sum_{k=0}^{n} F(k)P(N(t)=k)}. \end{aligned} \tag{5}$$

Analogous to the definition of *c*-IM, when a network is up, it is of interest to identify which path makes the network in the up state, and what is the contribution of edge *i* to the up state of the network. This motivates us to develop a new network IM as follows.

Definition 4: Consider a network with *n* edges. Let the occurrence of edge failures follow a counting process $\{N(t),\, t \geq 0\}$. The *p*-IM of edge *i*, denoted by $I^p(i,t)$, is the conditional probability that a path including edge *i* is up at time *t*, given that the network is up at time *t*. Mathematically, we have

$$\begin{aligned} I^p(i,t) &= P(\text{a path } P \in P_i \text{ is up at } t \,|\text{the network is up at } t) \\ &= \frac{P(\text{a path } P \in P_i \text{ is up at } t)}{P(\text{the network is up at } t)} \\ &= \frac{\sum_{k=0}^{n} P(N(t)=k)F'(n-k,1_i)}{\sum_{k=0}^{n} F'(n-k)P(N(t)=k)}. \end{aligned} \tag{6}$$

Remark 1: When the edge failures follow a counting process $\{N(t),\, t \geq 0\}$, the *c*-IM estimates the contribution of individual edges to network failure. However, the *p*-IM evaluates the contribution of individual edges to network functionality. Both IMs can indicate which edge has the greatest impact on network reliability. In addition, both *c*-IM and *p*-IM do not require specific probabilistic information of individual edge, yet they depend on the distribution of *N*(*t*) and the spectra of the network.

In the following, a simple example is given to illustrate basic notations.

Example 1: Consider a simple network shown in Fig. 1. There are two terminals $\{s,\, t\}$ and four edges. First, the following results are derived for edge 3, $k = 2$ and $n = 4$.

1) $C_3 = \{\{3,1\},\{3,2\},\{3,1,4\},\{3,2,4\},\{3,1,2\},\{3,1,2,4\}\}$, $P_3 = \{\{3,2,4\},\{3,1,2\},\{3,1,2,4\}\}$.
2) $C(2) = \{\{2,1\},\{4,2\},\{3,2\},\{3,1\},\{1,4\}\}$, $P(2) = \{\{2,1\}\}$.
3) $C_3(2) = \{\{3,2\},\{3,1\}\}$, $P_3(2) = \phi$.
4) $C_{(3)}(2) = \{\{1,2\},\{2,4\},\{1,4\}\}$, $P_{(3)}(2) = \{\{2,1\}\}$.
5) $F(2) = |C(2)|\binom{4}{2}^{-1} = \frac{5}{6}$, $F'(2) = |P(2)|\binom{4}{2}^{-1} = \frac{1}{6}$.
6) $F(2,0_3) = |C_3(2)|\binom{4}{2}^{-1} = \frac{1}{3}$, $F(2,1_3) = |C_{(3)}(2)|\binom{4}{2}^{-1} = \frac{1}{2}$.
7) $F'(2,0_3) = |P_{(3)}(2)|\binom{4}{2}^{-1} = \frac{1}{6}$, $F'(2,1_3) = |P_3(2)|\binom{4}{2}^{-1} = 0$.

Moreover, we can verify that the D-spectrum and C-spectrum of the network are {1/4, 5/6, 1, 1} and {0, 1/6, 3/4, 1}, respectively. Based on (1) and (3), we obtain

$$P(T \leq t) = \frac{1}{4}P(N(t)=1) + \frac{5}{6}P(N(t)=2) + P(N(t)=3) + P(N(t)=4)$$

and

$$P(T > t) = P(N(t)=0) + \frac{3}{4}P(N(t)=1) + \frac{1}{6}P(N(t)=2).$$

Finally, it is easy to verify that the D-spectrum and C-spectrum of edge 3 are {0, 1/3, 3/4, 1} and {0, 0, 1/2, 1}, respectively. Thus, from (5) and (6), the *c*-IM and *p*-IM of edge 3 are given as

$$I^c(3,t) = \frac{\frac{1}{3}P(N(t)=2) + \frac{3}{4}P(N(t)=3) + P(N(t)=4)}{\frac{1}{4}P(N(t)=1) + \frac{5}{6}P(N(t)=2) + P(N(t)=3) + P(N(t)=4)}$$

and

$$I^p(3,t) = \frac{P(N(t)=0) + \frac{1}{2}P(N(t)=1)}{P(N(t)=0) + \frac{3}{4}P(N(t)=1) + \frac{1}{6}P(N(t)=2)}$$

respectively.

B. Relation to Other Importance Measures

Hwang [17] proposed the H-IM to compare the structural importance of edges. That is, $i \geq_h j$ if $|C_i(k)| \geq |C_j(k)|$ for all *k*, where $i \geq_h j$ means that edge *i* is more important than edge *j*. The relation between H-IM and *c*-IM is given by the following theorem.

Theorem 1: Consider a network composed of *n* edges, and edge failures occur according to a counting process $\{N(t),\, t \geq 0\}$. If $i >_h j$, then $I^c(i,t) > I^c(j,t)$ for all *t*.

The Birnbaum structural IM is a special case of the Birnbaum IM with common edge reliability $p = 0.5$. The study in [13] shows that $i >_b j$ is equivalent to $|C_i| > |C_j|$, where $i >_b j$ implies that edge *i* is more important than edge *j*. The following theorem connects the *c*-IM and the Birnbaum structural IM.

Theorem 2: Consider a network with *n* edges, and edge failures occur according to a counting process $\{N(t),\, t \geq 0\}$. If $P(N(t)=k) = \binom{n}{k}A^{-1}$, $k = 1,2,\ldots,n-1$, where $A > \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1}$, then $I^c(i,t) > I^c(j,t)$ if and only if $i >_b j$.

Remark 2: The H-IM assumes that $|C_i(k)| \geq |C_j(k)|$ holds for all *k*. This assumption is too rigorous and only satisfied in a certain network structure. In reality, $|C_i(k)| \geq |C_j(k)|$ may hold only for certain *k*, but not all *k*. Hence, H-IM indeed represents a partial ranking because some edges may not be compared. However, the proposed *c*-IM is a complete ranking of all edges relative to their contribution to network failure. Moreover, Theorem 2 shows that *c*-IM is equivalent to the Birnbaum structural IM when *N*(*t*) follows a special distribution. In fact, *N*(*t*) may have arbitrary distribution. Then, the Birnbaum structural IM fails to identify the importance of edges, whereas the *c*-IM can.

In order to relate the *c*-IM to *p*-IM, the following lemma is presented.

Lemma 1: $F(k, 0_i) - F(k, 0_j) = F'(n-k, 1_i) - F'(n-k, 1_j)$.

Now, the following theorem shows that *c*-IM is equivalent to *p*-IM.

Theorem 3: $I^c(i,t) > I^c(j,t)$ if and only if $I^p(i,t) > I^p(j,t)$.

III. Importance Measures Under an SNHPP

This section discusses a special case where the failures of edges follow an SNHPP. Recall that a counting process $\{N(t), t \geq 0\}$ is a standard NHPP with intensity function $\lambda(t)$, if

1) $N(0) = 0$,
2) $N(t)$ has independent increment, namely, the numbers of failures in two disjoint time intervals are independent,
3) $P\{N(t+h) - N(t) \geq 2\} = o(h)$, and
4) $P\{N(t+h) - N(t) = 1\} = \lambda(t)h + o(h)$.

It can be shown that

$$P\left(N(t) = k\right) = \frac{(\Lambda(t))^k}{k!} e^{-\Lambda(t)}, \quad k = 0, 1, 2, \dots$$

where $\Lambda(t) = E(N(t)) = \int_0^t \lambda(t)dt$. The function $\Lambda(t)$ is called the mean value function.

Note: since $\lambda(t) > 0$, $\lim_{t\to 0} \int_0^t \lambda(t)dt = \lim_{t\to 0} \Lambda(t) = 0$, and $\lim_{t\to +\infty} \int_0^t \lambda(t)dt = \lim_{t\to +\infty} \Lambda(t) = +\infty$.

In fact, when $\lambda(t)$ is a constant number λ, the NHPP becomes a homogeneous Poisson process (HPP). For more details about the NHPP, one may refer to [27] and [28].

Note that in theory $N(t)$ can approach infinity as t increases. Since the maximum number of failed edges for a network is n, the counting process $N(t)$ indeed is an SNHPP. It is the same as NHPP with intensity function $\lambda(t)$, except that the process terminates when all n edges are down. The probability mass function for an SNHPP is given by

$$P(N(t) = k) = \begin{cases} \frac{(\Lambda(t))^k}{k!} e^{-\Lambda(t)}, & k = 0, 1, \dots, n-1, \\ 1 - \sum_{k=0}^{n-1} \frac{(\Lambda(t))^k}{k!} e^{-\Lambda(t)}, & k = n. \end{cases} \quad (7)$$

By combining (7) with the law of total probability, the following result holds:

$$P(T > t) = \sum_{k=0}^{n-1} \frac{(\Lambda(t))^k}{k!} e^{-\Lambda(t)} \overline{F}(k) \quad (8)$$

where $\overline{F}(k) = 1 - F(k)$, which is the probability that the network is up if k randomly chosen edges are down.

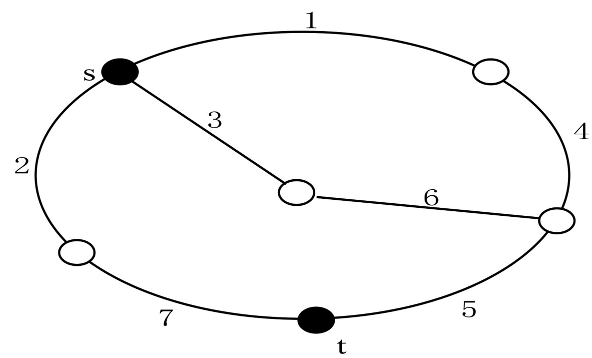

Fig. 2. Network with seven edges and two terminals.

Hence

$$P(T < t) = 1 - P(T > t) = 1 - \sum_{k=0}^{n-1} \frac{(\Lambda(t))^k}{k!} e^{-\Lambda(t)} \overline{F}(k). \quad (9)$$

Next, based on (7) and $F(n, 0_i) = 1$, the following results are obtained:

$$I^c(i,t) = \frac{\sum_{k=1}^{n-1} \frac{(\Lambda(t))^k}{k!} e^{-\Lambda(t)} F(k, 0_i) + 1 - \sum_{k=0}^{n-1} \frac{(\Lambda(t))^k}{k!} e^{-\Lambda(t)}}{1 - \sum_{k=0}^{n-1} \frac{(\Lambda(t))^k}{k!} e^{-\Lambda(t)} \overline{F}(k)} \quad (10)$$

and

$$I^p(i,t) = \frac{\sum_{k=0}^{n-1} \frac{(\Lambda(t))^k}{k!} e^{-\Lambda(t)} F'(n-k, 1_i)}{\sum_{k=0}^{n-1} \frac{(\Lambda(t))^k}{k!} e^{-\Lambda(t)} \overline{F}(k)}. \quad (11)$$

A. C-Importance Measure Under the SNHPP

Let $k_{\min\{i,j\}} = \min\{k : F(k, 0_i) \neq F(k, 0_j)\}$ and $k_{\max\{i,j\}} = \max\{k : F(k, 0_i) \neq F(k, 0_j)\}$. The following theorem characterizes the *c*-IM ranking when *t* is either sufficiently large or small.

Theorem 4: Consider a network consisting of *n* edges. Assume edge failures occur according to the SNHPP with intensity function $\lambda(t)$. Thus

i) Suppose that $F(k_{\min\{i,j\}}, 0_i) > F(k_{\min\{i,j\}}, 0_j)$. Then there exist t_0 such that for all $t \leq t_0$, the inequality $I^c(i,t) \geq I^c(j,t)$ holds.
ii) Suppose that $F(k_{\max\{i,j\}}, 0_i) > F(k_{\max\{i,j\}}, 0_j)$. Then, there exist t_0 such that for all $t \geq t_0$, the inequality $I^c(i,t) \geq I^c(j,t)$ holds.

Generally, the inequality $I^c(i,t) \geq I^c(j,t)$ depends on the network structure and time *t*. However, for large or small *t*, Theorem 4 shows the inequality $I^c(i,\lambda) \geq I^c(j,\lambda)$ depends only on the network structure through the D-spectrum of corresponding edge. The following example is used to illustrate the application of Theorem 4.

Example 2: Consider the network in Fig. 2. It has 7 edges, 6 nodes, and 2 terminals. All edges are subject to failure, and the network is up if and only if terminals *s* and *t* are connected by

TABLE I
D-SPECTRUM FOR NETWORK (INCLUDE EDGES 5 AND 7) IN FIG. 2

k	1	2	3	4	5	6	7
$F(k)$	0	2/21	17/35	4/5	20/21	1	1
$F(k,0_5)$	0	2/21	9/35	16/35	2/3	6/7	1
$F(k,0_7)$	0	1/21	9/35	18/35	5/7	6/7	1

operational edges. Table I presents the D-spectra of edges 5, 7 and the whole network for comparing the two edges based on the c-IM. Suppose the edge failures follow an SNHPP with intensity function $\alpha t^{\alpha-1}$, where $0 < \alpha < 1$. Note that the SNHPP has a decreasing intensity function.

Now, from (10), and $\int_0^t \alpha t^{\alpha-1} dt = \Lambda(t) = t^\alpha$, we have

$$I^c(5,t) = \frac{\frac{2}{21}\frac{t^{2\alpha}}{2!} + \frac{9}{35}\frac{t^{3\alpha}}{3!} + \frac{16}{35}\frac{t^{4\alpha}}{4!} + \frac{2}{3}\frac{t^{5\alpha}}{5!} + \frac{6}{7}\frac{t^{6\alpha}}{6!} + e^{t^\alpha} - \sum_{k=0}^{6}\frac{t^{k\alpha}}{k!}}{e^{t^\alpha} - 1 - t^\alpha - \frac{19}{21}\frac{t^{2\alpha}}{2!} - \frac{18}{35}\frac{t^{3\alpha}}{3!} - \frac{1}{5}\frac{t^{4\alpha}}{4!} - \frac{1}{21}\frac{t^{5\alpha}}{5!}}$$

and

$$I^c(7,t) = \frac{\frac{1}{21}\frac{t^{2\alpha}}{2!} + \frac{9}{35}\frac{t^{3\alpha}}{3!} + \frac{18}{35}\frac{t^{4\alpha}}{4!} + \frac{5}{7}\frac{t^{5\alpha}}{5!} + \frac{6}{7}\frac{t^{6\alpha}}{6!} + e^{t^\alpha} - \sum_{k=0}^{6}\frac{t^{k\alpha}}{k!}}{e^{t^\alpha} - 1 - t^\alpha - \frac{19}{21}\frac{t^{2\alpha}}{2!} - \frac{18}{35}\frac{t^{3\alpha}}{3!} - \frac{1}{5}\frac{t^{4\alpha}}{4!} - \frac{1}{21}\frac{t^{5\alpha}}{5!}}.$$

According to Table I, we observe that $k_{min\{5,7\}} = \min\{k : F(k,0_5) \neq F(k,0_7)\} = 2$, and $F(2,0_5) = \frac{2}{21} > F(2,0_7) = \frac{1}{21}$. Thus, we obtain $I^c(5,t) > I^c(7,t)$ for small enough t using Theorem 4(i).

Similarly, using Theorem 4(ii), we obtain $I^c(5,t) < I^c(7,t)$ for sufficiently large t followed from the fact that $k_{\max\{5,7\}} = \max\{k : F(k,0_5) \neq F(k,0_7)\} = 5$, and $F(5,0_5) = \frac{2}{3} < F(5,0_7) = \frac{5}{7}$, which can be observed in Table I.

Let $k_i = \min\{k : F(k,0_i) \neq 0\}$ and $k_j = \min\{k : F(k,0_j) \neq 0\}$. When t is small enough, the following theorem shows that edges can be compared by only determining the quantities k_i and $F(k_i,0_i)$.

Theorem 5: Consider a network that consists of n edges. Assume that the edge failures occur in an SNHPP with a intensity function $\lambda(t)$. Thus

i) If $k_i > k_j$, then there exists t_0 such that for all $t \leq t_0$, the inequality $I^c(i,t) < I^c(j,t)$ holds.
ii) If $k_i = k_j$ and $F(k_i,0_i) > F(k_j,0_j)$, then there exists t_0 such that for all $t \leq t_0$, the inequality $I^c(i,t) > I^c(j,t)$ holds.

Let $a = \min\{k: F(k) \neq 0\}$, then the following theorem shows edges can be compared by determining $F(a,0_i)$ when t is small enough.

Theorem 6: Assume that the failures of edges occur in an SNHPP with intensity function $\lambda(t)$ for a network with n edges. If $F(a,0_i) > F(a,0_j)$, there exist t_0 such that for all $t \leq t_0$, the inequality $I^c(i,t) > I^c(j,t)$ holds.

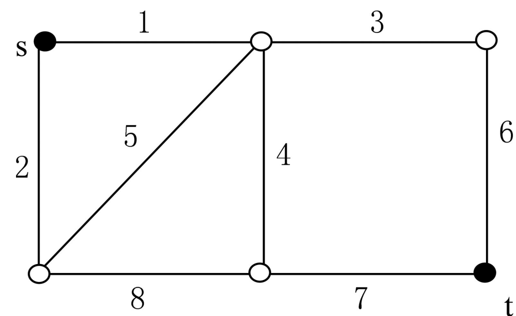

Fig. 3. Network with eight edges and two terminals.

Remark 3: Noting that $F(a,i) = |C_i(a)|\binom{n}{a}^{-1}$, $i = 1, 2, \ldots, n$, $F(a,0_i) > F(a,0_j)$ is equivalent to $|C_i(a)| > |C_j(a)|$. Recall that the first-term IM of edge i is defined as $|C_i(a)|$ [29]. Theorem 6 shows that the c-IM ranking agrees with the first-term ranking in the case where t is small enough. However, in the case for general t, the first-term IM fails to identify the importance of edge, whereas the c-IM can be applicable to this case.

Theorems 5 and 6 can be interpreted by the following example.

Example 3: Consider the network with 8 edges, 6 nodes, and 2 terminals shown in Fig. 3. Edges are subject to failure, and the network is down whenever two terminal are disconnected. Assume edge failures follow an SNHPP with a decreasing intensity function $\alpha t^{\alpha-1}$ for $0 < \alpha < 1$.

It can be seen that the smallest cut size is 2. Thus, $a = 2 = \min\{k : F(k) \neq 0\}$. In addition, $C_3(2) = \{\{3,7\}\}, C_7(2) = \{\{3,7\},\{6,7\}\}$. Hence

$$F(2,0_3) = |C_3(2)|\binom{8}{2}^{-1} < F(2,0_7) = |C_7(2)|\binom{8}{2}^{-1}.$$

Using Theorem 6, if t is small enough, we conclude that $I^c(3,t) < I^c(7,t)$ from the fact $F(2,0_3) < F(2,0_7)$.

Moreover, from Fig. 3, the size of the smallest cut including edge 4 is 3. Hence, $k_4 = 3 = \min\{k : F(k,0_4) \neq 0\}$. However, $k_3 = 2 = \min\{k : F(k,0_3) \neq 0\}$. Thus, using Theorem 5, we obtain $I^c(3,t) > I^c(4,t)$ for small enough t followed from the fact $k_4 > k_3$.

From $\lim_{t\to+\infty}\int_0^t \lambda(t)dt = \lim_{t\to+\infty}\Lambda(t) = +\infty$, we have $\lim_{t\to\infty}\frac{(\Lambda(t))^m}{e^{\Lambda(t)}} = 0$ for all $m \geq 0$. Thus, by the definition of $I^c(i,t)$, it can be seen that $\lim_{t\to\infty} I^c(i,t) = 1$. Hence, $\lim_{t\to\infty}\frac{I^c(i,t)}{I^c(j,t)} = 1$. Therefore, $I^c(i,t)$ and $I^c(j,t)$ have nearly

TABLE II
C-SPECTRUM FOR NETWORK (INCLUDE EDGES 2 AND 5) IN FIG. 2

k	1	2	3	4	5	6	7
$F'(k)$	0	1/21	7/35	18/35	19/21	1	1
$F'(k,1_2)$	0	1/21	1/7	12/35	2/3	6/7	1
$F'(k,1_5)$	0	0	3/35	12/35	5/7	6/7	1

equal values when t is large enough. However, the inequality $I^c(i,t) > I^c(j,t)$ can be tested on the condition of part (ii) of Theorem 4 when time t is large enough.

B. P-Importance Measure Under the SNHPP

In this subsection, the p-IM of the network edges is investigated when the failures of edges follow an SNHPP. Since the results here are similar to those of the preceding subsection, the detailed proofs are omitted.

Let $k'_{\min\{i,j\}} = \min\{k : F'(n-k, 1_i) \neq F'(n-k, 1_j)\}$, $k'_{\max\{i,j\}} = \max\{k : F'(n-k, 1_i) \neq F'(n-k, 1_j)\}$, then the following theorem is established, which is similar to Theorem 4.

Theorem 7: Consider a network with n edges. Assume that the failures of edges follow an SNHPP with a intensity function $\lambda(t)$. Thus

i) Suppose that $F'(n-k'_{\min\{i,j\}}, 1_i) > F'(n-k'_{\min\{i,j\}}, 1_j)$. Then there exists t_0, such that for all $t \leq t_0$, the inequality $I^p(i,t) \geq I^p(j,t)$ holds.

ii) Suppose that $F'(n-k'_{\max\{i,j\}}, 1_i) > F'(n-k'_{\max\{i,j\}}, 1_j)$. Then there exists t_0 such that for all $t \geq t_0$, the inequality $I^p(i,t) \geq I^p(j,t)$ holds.

Similar to c-IM, the inequity $I^p(i,t) \geq I^p(j,t)$ depends on the structure of the network and time t. However, Theorem 7 shows that p-IM ranking depends only on the structure of the network through the C-spectrum of corresponding edge in the case where t is either sufficiently large or small.

The following example demonstrates the application of Theorem 7.

Example 4: Consider the network in example 2. Suppose the failures of edges follow an SNHPP with intensity function $\alpha t^{\alpha-1}$, where $0 < \alpha < 1$. The C-spectra of edges 2, 5 and the entire network are shown in Table II for comparing the two edges based on the p-IM.

Using (11), we obtained that

$$I^p(2,t) = \frac{1 + \frac{6}{7}t^\alpha + \frac{2}{3}\frac{t^{2\alpha}}{2!} + \frac{12}{35}\frac{t^{3\alpha}}{3!} + \frac{1}{7}\frac{t^{4\alpha}}{4!} + \frac{1}{21}\frac{t^{5\alpha}}{5!}}{1 + t^\alpha + \frac{19}{21}\frac{t^{2\alpha}}{2!} + \frac{18}{35}\frac{t^{3\alpha}}{3!} + \frac{1}{5}\frac{t^{4\alpha}}{4!} + \frac{1}{21}\frac{t^{5\alpha}}{5!}}$$

and

$$I^p(5,t) = \frac{1 + \frac{6}{7}t^\alpha + \frac{5}{7}\frac{t^{2\alpha}}{2!} + \frac{12}{35}\frac{t^{3\alpha}}{3!} + \frac{3}{35}\frac{t^{4\alpha}}{4!}}{1 + t^\alpha + \frac{19}{21}\frac{t^{2\alpha}}{2!} + \frac{18}{35}\frac{t^{3\alpha}}{3!} + \frac{1}{5}\frac{t^{4\alpha}}{4!} + \frac{1}{21}\frac{t^{5\alpha}}{5!}}.$$

In Table II, we observe that $k'_{\min\{2,5\}} = \min\{k : F'(7-k, 1_2) \neq F'(7-k, 1_5)\} = 2$, and $k'_{\max\{2,5\}} = \max\{k : F'(7-k, 1_2) \neq F'(7-k, 1_5)\} = 5$. Thus, using Theorem 7, we obtain that $I^p(2,t) > I^p(5,t)$ for sufficiently large t from the fact $F'(7-5, 1_5) = 0 < F'(7-5, 1_2) = \frac{1}{21}$. However, for sufficiently small t, $I^p(2,t) < I^p(5,t)$ from the fact $F'(7-2, 1_5) = \frac{5}{7} > F'(7-2, 1_2) = \frac{2}{3}$.

Let $k'_i = \min\{k : F'(k, 1_i) \neq 0\}$ and $k'_j = \min\{k : F'(k, 1_i) \neq 0\}$. Similar to Theorem 5, the following theorem is presented to characterize the p-IM ranking.

Theorem 8: Consider a network with n edges. Assume that the edge failures occur in an SNHPP with a intensity function $\lambda(t)$. Thus

i) If $k'_i < k'_j$, then there exists t_0 such that for all $t \geq t_0$, the inequality $I^p(i,t) > I^p(j,t)$ holds.

ii) If $k'_i = k'_j$ and $F'({k_i}', 1_i) > F'({k_j}', 1_j)$, then there exists t_0 such that for all $t \geq t_0$, the inequality $I^p(i,t) > I^p(j,t)$ holds.

Let $a' = \min\{k : F'(k) \neq 0\}$. Similar to Theorem 6, the following theorem can be established in the case of where the time t is large enough.

Theorem 9: Assume that the failures of edges occur in an SNHPP with intensity function $\lambda(t)$ for a network with n edges. Thus

If $F'(a', 1_i) > F'(a', 1_j)$, then there exists t_0 such that for all $t \geq t_0$, the inequality $I^p(i,t) > I^p(j,t)$ holds.

Remark 4: Note that $F'(a', 1_i) = |P_i(a')|\binom{n}{a'}^{-1}$, $i = 1, 2, \ldots, n$, the condition in Theorem 9 is equivalent to $|P_i(a')| > |P_j(a')|$. Let us recall the definition of the rare-event IM of edge i, which is defined as $|P_i(a')|$ [29]. Based on the rare-event IMs, edge i is more important than edge j if and only if $|P_i(a')| > |P_j(a')|$. Analogous to the first-term IMs, Theorem 9 shows that the p-IM ranking agrees with the ranking induced by the rare-event IM in the case that t is large enough. However, in the case for general t, the rare-event IM fails to identify the importance of edge, whereas the p-IM can be applicable to this case.

Theorems 8 and 9 can be explained by the following example.

Example 5: Consider the network in example 3. Suppose the failures of edges follow an SNHPP with intensity function $\alpha t^{\alpha-1}$, where $0 < \alpha < 1$. Fig. 3 reveals that the size of the smallest path is 3. Thus, $a' = 3 = \min\{k : F'(k) \neq 0\}$. In addition, Fig. 3 indicates that $P_1(3) = \{\{1,3,6\},\{1,4,7\}\}$, $P_2(3) = \{\{2,8,7\}\}$. Thus

$$F'(3, 1_1) = |P_1(3)|\binom{8}{2}^{-1} > F'(3, 1_2) = |P_2(3)|\binom{8}{2}^{-1}.$$

This article has been accepted for inclusion in a future issue of this journal. Content is final as presented, with the exception of pagination.

TABLE III
D-SPECTRUM FOR EDGE OF THE SIMPLE NETWORK

k	$F(k,0_1)$	$F(k,0_2)$	$F(k,0_3)$	$F(k,0_4)$
1	0	1/4	0	0
2	1/2	1/2	1/3	1/3
3	3/4	3/4	3/4	3/4
4	1	1	1	1

TABLE IV
C-SPECTRUM FOR EDGE OF THE SIMPLE NETWORK

k	$F'(k,1_1)$	$F'(k,1_2)$	$F'(k,1_3)$	$F'(k,1_4)$
1	0	0	0	0
2	1/6	1/6	0	0
3	1/2	3/4	1/2	1/2
4	1	1	1	1

From Theorem 9, $I^p(1,t) > I^p(2,t)$ for sufficiently large t.

Moreover, from Fig. 3, the size of the smallest path including edge 5 is 4. Hence, $k_5 = 4 = \min\{k : F'('(k,1_5) \neq 0\}$. However, from $P_2(3) = \{\{2,8,7\}\}, k_2 = 3 = \min\{k : F'(k,1_2) \neq 0\}$. Thus, using Theorem 8, we obtain $I^p(2,t) > I^p(5,t)$ for sufficiently large t followed from the fact $k_5 > k_2$.

From the continuity of $I^p(i,t)$ at zero, we have $\lim_{t\to 0} I^p(i,t) = 1$. Hence, $\lim_{t\to 0} \frac{I^p(i,t)}{I^p(j,t)} = 1$. Therefore, $I^p(i,t)$ and $I^p(j,t)$ have nearly equal values in the case where the time t is small enough. However, the inequality $I^p(i,t) > I^p(j,t)$ should be verified in terms of the condition in part (i) of Theorem 7.

IV. NUMERICAL EXAMPLES

In this section, two numerical examples are provided to illustrate how the proposed IMs can effectively assist in identifying the critical edges with respect to the reliability of the network. Assume that the failures of edges occur according to an SNHPP with intensity function $\lambda(t)$.

A. Simple Network

Example 6: Consider the simple network in Fig. 1. The D-spectrum and C-spectrum for the edges are shown in Tables III and IV, respectively. To obtain an accurate result, the calculations are performed using the definition of the spectra.

From the results shown in Table III, we find that $F(k,0_2) \geq F(k,0_1) \geq F(k,0_3) = F(k,0_4)$ for all $k = 1$, 2, 3, 4. By the definition of D-spectrum of edge, that is, $F(k,0_i) = |C_i(k)|\binom{n}{k}^{-1}$, we have $|C_2(k)| \geq |C_1(k)| \geq |C_3(k)| = |C_4(k)|$ for all $k = 1,2,3,4$. Hence, $2 \geq_h 1 \geq_h 3 =_h 4$ are obtained from the definition of H-IM [17]. Furthermore, using Theorem 1, $I^c(2,t) > I^c(1,t) > I^c(3,t) = I^c(4,t)$ for all t. As for p-IM, similar observations can be made from Table IV.

For two arbitrary fixed edges i and j, this simple network reveals that $F(k,0_i) \geq F(k,0_j)$ or $F(k,0_i) \leq F(k,0_j)$ for all k. However, this result is not always true. In the following transportation network, we will see that there exists edges i and $j, F(k,0_i) \geq F(k,0_j)$ for some k, whereas $F(k,0_i) \leq F(k,0_j)$ for other k.

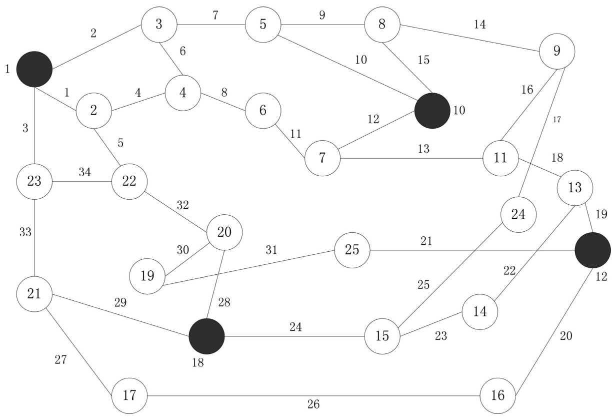

Fig. 4. Transportation network with 25 nodes, 34 edges, and 4 terminals.

B. Transportation Network

Example 7: Fig. 4 shows a four-terminal transportation network with 25 nodes and 34 edges. This network is originally in [24, Section 2.3]. Indeed exact calculation of the spectrum is an NP-hard issue. It is thus intractable for medium and large size network. Therefore, a feasible approach is to use Monte Carlo (MC) simulation to approximate the spectrum.

Gertsbakh and Shpungin [22] proposed the MC algorithm to obtain the estimates for the C-spectrum. The algorithm works as follows. To estimate $F'(k)$ and $F'(k,1_i)$, we simulate M random permutations of edge number. For each permutation, we start a sequential construction process of edges along the permutation from left to right, until the network enters the up state. Here, we count the number a_k of such permutations that the network is up when the exactly first k edges are up, and the number $b_{k,i}$ of such permutations that the network is up when the exactly first k edges are up, and edge i is among these k edges. Finally, we take the quantity $\hat{F}'(k) = \frac{a_k}{M}$ and $\hat{F}'(k,1_i) = \frac{b_{k,i}}{M}$ as the estimate of $F'(k)$ and $F'(k,1_i)$, respectively. Note that there exist 34! permutations in theory. However, it is impossible to generate all the permutations from the computation standpoint. Thus, we take value $M = 1\,000\,000$. In the following, the algorithmic is described in details.

1) Set all a_k and $b_{k,i}$ to be $0, k = 1,\ldots,n;\ i = 1,\ldots,n$.
2) Randomly generate a permutation $\pi \in \Pi_E$. (Π_E is the set of all edge permutations.)
3) Find the minimum index of the edge $r = r(\pi)$ such that the first r edges in π ensure network up state.
4) Set $a_r = a_r + 1$.

TABLE V
RANKINGS OBTAINED WITH THE c-IM FOR $t = 0.015$, 7, AND 30

edge i	$I^c(i, 0.015)$	edge i	$I^c(i, 7)$	edge i	$I^c(i, 30)$	edge i	First-term
19	0.6877447	19	0.4653227	30	0.99731364	19	9
31	0.2319925	7	0.4562961	15	0.99731356	31	3
20	0.2310432	18	0.4547820	2	0.99731317	20	3
26	0.2310019	2	0.4523352	8	0.99731274	26	3
30	0.2285656	24	0.4512400	6	0.99731250	30	3
21	0.2284245	3	0.4508477	7	0.99731234	21	3
27	0.2269074	29	0.4507045	33	0.99731201	27	3
12	0.1543193	33	0.4496841	16	0.99731195	12	2

TABLE VI
RANKINGS OBTAINED WITH THE p-IM FOR $t = 0.015$, 7, AND 30

edge i	$I^p(i, 0.015)$	edge i	$I^p(i, 7)$	edge i	$I^p(i, 30)$	edge i	Rare-event
18	0.99694369	19	0.80167210	7	0.74310764	7	1
12	0.99694362	7	0.75878373	2	0.73416167	2	1
8	0.99694293	18	0.75256077	3	0.69921400	3	1
11	0.99694251	2	0.74042601	19	0.69921235	29	1
31	0.99694169	24	0.73609987	29	0.66932982	33	1
6	0.99694094	3	0.73446654	33	0.65361871	10	1
24	0.99694093	29	0.73315229	10	0.63327805	20	1
7	0.99693856	33	0.72827585	18	0.61151398	26,27	1

5) Look for all i such that e_i resides in one of the first r positions in π, and for each i, set $b_{r,i} = b_{r,i} + 1$.
6) Set $r = r + 1$. If $r \le n$, go to step 4.
7) Repeat steps 2–6 M times.
8) Estimate $F'(k)$ and $F'(k, 1_i)$ through $\hat{F}'(k) = \frac{a_k}{M}$ and $\hat{F}'(k, 1_i) = \frac{b_{k,i}}{M}$, respectively.

Once the estimates for $F'(k)$ and $F'(k, 1_i)$ are available, we can obtain the estimates for c-IM and p-IM from (10) and (11). The above-mentioned algorithm can be modified to obtain the estimates of the D-spectrum [24].

Suppose the failures of edges follow an SNHPP with intensity function $\lambda(t) = \beta\alpha t^{\alpha-1}$ for $\alpha > 0$ and $\beta > 0$. The intensity function $\lambda(t)$ decreases in time t for $0 < \alpha < 1$, remains constant in t for $\alpha = 1$, and increases in t for $1 < \alpha < \infty$. In our practical case, when all edges are up, $\lambda(t)$ should be higher with respect to a situation where k edges are already down. Thus, we assume that intensity function $\lambda(t)$ decreases in t, i.e., $0 < \alpha < 1$.

Let $\Lambda(t) = \int_0^t \beta\alpha t^{\alpha-1} dt = \beta t^{\alpha} = 3t^{0.8} (\beta = 3, \alpha = 0.8)$. Based on (10) and (11), the evaluations of c-IM and p-IM are performed with the MC simulation method described in Gertsbakh and Shpungin [22], [24]. In order to present a measure of statistical robustness with respect to the estimated values of the c-IM and p-IM, ten independent simulations have been performed. Based on one million randomly generated network edge permutations, each simulation run is able to obtain an estimate for each new IM at three different time instants. It turns out that the top eight edges are almost consistent with each other, i.e., there exist at most two different edges comparing two different simulation results based on the same measure and time t. Hence, the MC approximations are robust in terms of the importance ranking.

From (10) and (11), the evaluations of the c-IM and p-IM come down to the estimations of the C-spectrum and D-spectrum, respectively. For the network (see Fig. 4) with 34 edges, we obtained the estimations of C-spectra (or D-spectra) using a microcomputer in less than one hour when taking $M = 1\,000\,000$ for each simulation. Recall that the spectra are structure invariants, which depend only on the network structure. Thus, for any given network, they are estimated once and for all. Therefore, the computational time for the spectra using the above-mentioned MC method is not an important issue. This fact illustrates that the practicality of the proposed measures as decisions often needs to be made in real time when network edges are failing.

Tables V and VI present the edge rankings according to c-IM and p-IM under $t = 0.015$, 7, and 30. For all cases, only eight most important edges are reported. Moreover, the results of edge rankings according to first-term IM and rare-event IM are included in Tables V and VI, respectively. In fact, for the transportation network, the smallest cardinality of minimal paths

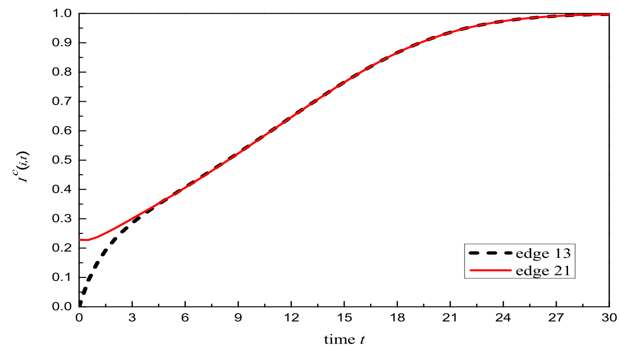

Fig. 5. Comparison of c-IM $I^c(i,t)$ between edges 13 and 21.

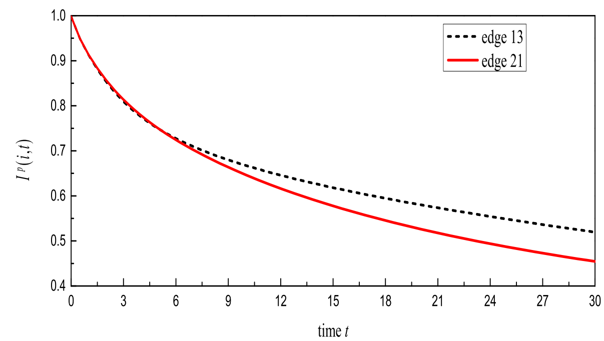

Fig. 6. Comparison of p-IM $I^p(i,t)$ between edges 13 and 21.

and cuts is 9 and 3, respectively. Thus, from Fig. 4, the first-term IMs of the eight most important edges are $|C_{19}(3)| = 9, |C_i(3)| = 3\,(i = 31,\ 20,\ 26,\ 30,\ 21,\ 27)$, and $|C_{12}(3)| = 2$. Moreover, a path $P = \{7, 2, 3, 29, 33, 10, 20, 26, 27\}$ is the unique minimal path of size 9. Hence, according to the rare-event IM, $|P_i(9)| = 1, \quad i \in P$ and $|P_i(9)| = 0, \quad i \notin P$. Therefore, based on rare-event IM, these nine edges, which belong to the minimal path P, are the most important ones.

In Tables V and VI, the importance rankings of edges for different times do not agree based on c-IM (p-IM). Moreover, from the first and seventh columns in Table V, the c-IM ranking for small t ($t = 0.015$) is in agreement with the first-term IM ranking. This result accords with Theorem 6. Similarly, from the fifth and seventh columns in Table VI, the rare-event IM ranking is quite consistent with the p-IM ranking for large t ($t = 30$). This observation is consistent with Theorem 9.

From Theorem 3, the ranking provided by c-IM and p-IM should be consistent given the same time t. However, except for $t = 7$, the ranking provided by these two measures is not consistent at the same time ($t = 0.015$ and 30) by comparing Tables V and VI. The reasons are interpreted as follows. From the sixth column in Table V, the c-IM of all edges at $t = 30$ is very close to each other, and all almost equals 1. Hence, it is difficult to identify which edges are more important based on c-IM. However, if one looks at the sixth column in Table VI, significant differences are observed for the p-IM values at $t = 30$. Thus, it is easier to identify the most important edges based on the p-IM. Therefore, as t becomes large, p-IM is more useful than c-IM. Similarly, comparing the second column in Tables V and VI, the p-IM at $t = 0.015$ is almost equal and close to 1, whereas the c-IM values differ significantly. Hence, if t is small, c-IM is more useful than p-IM to identity the most important edges.

Let us investigate the dependence of $I^c(i,t)$ and $I^p(i,t)$ on t. We choose edges 13, 21, and $\Lambda(t) = 3t^{0.8}$. The corresponding curves, shown in Figs. 5 and 6, are obtained based on (10) and (11), respectively.

Fig. 5 shows that for t close to 0, the c-IM of edges 13 and 21 tends toward 0 and 0.23, respectively. Moreover, as t approaches infinity, the c-IM of edges 13 and 21 all approach one. It is interesting to observe that for $t \in [0, 6]$, edge 21 is more important than edge 13, but for $t \in (6, 30]$, edges 21 and 13 become almost equally important. However, in Fig. 6, based on p-IM, edge 13 is more important than edge 21 for $t \in [6, 30]$, whereas for $t \in (0, 6]$, edges 21 and 13 become almost equally important. In summary, using Theorem 3, $I^c(21,t) > I^c(13,t)$ for $t \in (0, 6]$, whereas $I^c(21,t) < I^c(13,t)$ for $t \in [6, 30]$. The comparisons in Figs. 5 and 6 further confirm that for small t, the c-IM has a better capability to identify which edges are more important to the network failure. However, as t becomes large, the p-IM becomes more powerful to determine which edges are more important to the network operation.

V. Conclusion

In this paper, c-IM and p-IM are proposed to estimate the criticality of edges for multiterminal networks where failures of edges follow a counting process. The value of both IMs depends on the network structure and the distribution of the number of failed edges at a particular time, not on the lifetime distribution of the edges. Numerical experiments show that the c-IM can provide better ranking than the p-IM when t is relatively small. However, when t becomes large, the p-IM can lead to a better ranking than the c-IM. We also proved that both types of IMs provide consistent rankings regardless of their different definitions. When networks have special structures or the number of the failed edge follows certain special distributions, their rankings also agree with the results generated by existing IMs. In particular, when edges fail according to an SNHPP, the importance rankings depend only on the network structure as time becomes sufficiently large or small. The ranking results can be potentially used to prioritize the resource allocation in network design and maintenance. As one limitation, the proposed IMs rely on the distribution of the number of failed edges at a particular time, yet the lifetime of the individual edge is not considered. Our future research will incorporate the lifetime distribution at individual edge level for constructing more comprehensive IMs.

Appendix

Proof of Theorem 1: By Definition 1, and noting that $F(k, 0_i) = |C_i(k)|\binom{n}{k}^{-1}$

$$I^c(i,t) - I^c(j,t) =$$

$$\frac{\sum_{k=1}^{n} \binom{n}{k}^{-1} P(N(t) = k)\,(|C_i(k)| - |C_j(k)|)}{P(\text{the network is down at } t)}.$$

Hence, $I^c(i,t) > I^c(j,t)$ if and only if $\sum_{k=1}^{n}\binom{n}{k}^{-1} P(N(t)=k)(|C_i(k)| - |C_j(k)|) > 0$. This is clearly implied by the condition $|C_i(k)| \geq |C_j(k)|$ for all k. From the definition of H-IM, the theorem follows.

Proof: of Theorem 2: By the definition of $I^c(i,t)$ and $F(n,0_i) = F(n,0_j) = 1$

$$I^c(i,t) - I^c(j,t) = \frac{\sum_{k=1}^{n} P(N(t)=k)\left(F(k,0_i) - F(k,0_j)\right)}{P(\text{the network is down at } t)}$$

$$= \frac{\sum_{k=1}^{n-1} \binom{n}{k} A^{-1}\left(|C_i(k)| - |C_j(k)|\right)\binom{n}{k}^{-1}}{P(\text{the network is down at } t)}$$

$$= \frac{\sum_{k=1}^{n-1} A^{-1}\left(|C_i(k)| - |C_j(k)|\right)}{P(\text{the network is down at } t)}$$

$$= \frac{A^{-1}\left(|C_i| - |C_j|\right)}{P(\text{the network is down at } t)}$$

where the last equality is obtained from the fact that

$$|C_i| = |C_i(1)| + |C_i(2)| + \cdots + |C_i(n)| \text{ and } |C_i(n)| = |C_j(n)| = 1, \quad i = 1,2,\ldots,n.$$

Thus, $I^c(i,t) > I^c(j,t)$ is equivalent to $|C_i| > |C_j|$. In addition, $i >_b j$ is equivalent to $|C_i| > |C_j|$ [13]. Hence, the theorem is proved.

Proof: of Lemma 1: the following result is given due to [13], that is

$$|C_i(k)| + \left|P_{(i)}(n-k)\right| = \binom{n-1}{k-1}.$$

Thus

$$F(k,0_i) - F(k,0_j) = |C_i(k)|\binom{n}{k}^{-1} - |C_j(k)|\binom{n}{k}^{-1}$$

$$= \binom{n}{k}^{-1}\left[\left(\binom{n-1}{k-1} - \left|P_{(i)}(n-k)\right|\right) - \left(\binom{n-1}{k-1} - \left|P_{(j)}(n-k)\right|\right)\right]$$

$$= \binom{n}{k}^{-1}\left[\left|P_{(j)}(n-k)\right| - \left|P_{(i)}(n-k)\right|\right]$$

$$= \binom{n}{k}^{-1}\left[(|P(n-k)| - |P_j(n-k)|) - (|P(n-k)| - |P_i(n-k)|)\right]$$

$$= \binom{n}{n-k}^{-1}\left[|P_i(n-k)| - |P_j(n-k)|\right]$$

$$= F'(n-k,1_i) - F'(n-k,1_j).$$

Hence, the lemma follows.

Proof: of Theorem 3: By Definition 1

$$I^c(i,t) - I^c(j,t) = \frac{\sum_{k=1}^{n} P(N(t)=k)(F(k,0_i) - F(k,0_j))}{P(\text{the network is down at } t)}.$$

On the other hand, by Definition 2

$$I^p(i,t) - I^p(j,t) = \frac{\sum_{k=1}^{n} P(N(t)=k)(F'(n-k,1_i) - F'(n-k,1_j))}{\sum_{k=1}^{n} F'(n-k)P(N(t)=k)}.$$

Thus, from Lemma 1, the theorem follows.

Proof of Theorem 4: By definition of $I^c(i,\lambda)$

$$I^c(i,t) - I^c(j,t) = \frac{\sum_{k=1}^{n} P(N(t)=k)\left(F(k,0_i) - F(k,0_j)\right)}{P(\text{the network is down at } t)}$$

$$= \frac{\sum_{k=1}^{n-1} \frac{\Lambda(t)^k}{k!} e^{-\Lambda(t)}\left(F(k,0_i) - F(k,0_j)\right)}{P(\text{the network is down at } t)}$$

where $\Lambda(t) = E(N(t)) = \int_0^t \lambda(t)dt$ and the second equality is established from the fact that $F(n,0_i) = F(n,0_j) = 1$. Then $I^c(i,t) \geq I^c(j,t)$ if and only if

$$\sum_{k=1}^{n-1} \frac{\Lambda(t)^k}{k!}\left(F(k,0_i) - F(k,0_j)\right) \geq 0. \tag{12}$$

Thus, we have the following two cases.

1) From the definition of $k_{\min\{i,j\}}$ and (12), we obtain that $I^c(i,t) \geq I^c(j,t)$ is equivalent to $\sum_{k=k_{\min\{i,j\}}}^{n-1} \frac{\Lambda(t)^k}{k!}(F(k,0_i) - F(k,0_j)) \geq 0$. From $\lim_{t\to 0}\int_0^t \lambda(t)dt = \lim_{t\to 0}\Lambda(t) = 0$, we have $\frac{\Lambda(t)^k}{k!}(F(k,0_i) - F(k,0_j))$ tend to 0 as $t \to 0$, and more large the k is, more quickly it tends to zero. Hence, $\frac{\Lambda(t)^{k_{\min\{i,j\}}}}{k_{\min\{i,j\}}!}(F(k_{\min\{i,j\}},0_i) - F(k_{\min\{i,j\}},0_j))$ is the dominant term in the $\sum_{k=k_{\min\{i,j\}}}^{n-1}\frac{\Lambda(t)^k}{k!}(F(k,0_i) - F(k,0_j))$. Therefore, $I^c(i,t) \geq I^c(j,t)$ follows from the fact $F(k_{\min\{i,j\}},0_i) > F(k_{\min\{i,j\}},0_j)$.
2) From the definition of $k_{\max\{i,j\}}$ and (12), we obtain that $I^c(i,t) \geq I^c(j,t)$ is equivalent to $\sum_{k=1}^{k_{\max\{i,j\}}} \frac{\Lambda(t)^k}{k!}(F(k,0_i) - F(k,0_j)) \geq 0$. From $\lim_{t\to\infty}\int_0^t \lambda(t)dt = \lim_{t\to\infty}\Lambda(t) = \infty$, we have $\frac{\Lambda(t)^k}{k!}(F(k,0_i) - F(k,0_j))$ tend to ∞ as $t \to \infty$, and more large the k is, more quickly it tends to ∞. Hence, $\frac{\Lambda(t)^{k_{\max\{i,j\}}}}{k_{\max\{i,j\}}!}(F(k_{\max\{i,j\}},0_i) - F(k_{\max\{i,j\}},0_j))$ is the dominant term in the $\sum_{k=1}^{k_{\max\{i,j\}}}\frac{\Lambda(t)^k}{k!}(F(k,0_i) - F(k,0_j))$. Therefore, $I^c(i,t) \geq I^c(j,t)$ follows from the fact $F(k_{\max\{i,j\}},0_i) > F(k_{\max\{i,j\}},0_j)$.

Proof of Theorem 5: By the fact that $F(1,0_i) \leq F(2,0_i) \leq \cdots \leq F(n,0_i) = 1, i = 1,2,\ldots,n$, and from the definition of $k_i, F(k,0_i) = 0$ for all $k < k_i$. Similarly, $F(k,0_j) = 0$ for all $k < k_j$. Thus, based on (10), it can be seen

$$\lim_{t\to 0}\frac{I^c(i,t)}{I^c(j,t)}=\lim_{t\to 0}\frac{e^{\Lambda(t)}-1-\Lambda(t)-\frac{\Lambda(t)^2}{2!}-\cdots-\frac{\Lambda(t)^{k_i-1}}{(k_i-1)!}-\frac{\Lambda(t)^{k_i}}{k_i!}(1-F(k_i,0_i))}{e^{\Lambda(t)}-1-\Lambda(t)-\frac{\Lambda(t)^2}{2!}-\cdots-\frac{\Lambda(t)^{k_j-1}}{(k_j-1)!}-\frac{\Lambda(t)^{k_j}}{k_j!}(1-F(k_j,0_j))} \tag{14}$$

$$\lim_{t\to 0}\frac{I^c(i,t)}{I^c(j,t)}=\lim_{t\to 0}\frac{e^{\Lambda(t)}-1-\Lambda(t)-\cdots-\frac{\Lambda(t)^{k_i-k_j+1}}{(k_i-k_j+1)!}-\frac{\Lambda(t)^{k_i-k_j}}{(k_i-k_j)!}(1-F(k_i,0_i))}{e^{\Lambda(t)}-1+F(k_j,0_j))}=0<1$$

that

$$\frac{I^c(i,t)}{I^c(j,t)}=\frac{\sum_{k=k_i}^{n-1}\frac{\Lambda(t)^k}{k!}F(k,0_i)+e^{\Lambda(t)}-\sum_{m=0}^{n-1}\frac{\Lambda(t)^m}{m!}}{\sum_{k=k_j}^{n-1}\frac{\Lambda(t)^k}{k!}F(k,0_j)+e^{\Lambda(t)}-\sum_{m=0}^{n-1}\frac{\Lambda(t)^m}{m!}}.$$

For arbitrary positive integer *m*, from $\lim_{t\to 0}\int_0^t\lambda(t)dt=\lim_{t\to 0}\Lambda(t)=0$

$$\lim_{t\to 0}\frac{e^{\Lambda(t)}-1-\Lambda(t)-\frac{\Lambda(t)^2}{2!}-\cdots-\frac{\Lambda(t)^{k-1}}{(k-1)!}}{\Lambda(t)^m}=\begin{cases}0 & m<k;\\ \frac{1}{k!} & m=k;\\ +\infty & m>k.\end{cases} \tag{13}$$

Hence, we have (14) shown at the top of this page.

Repeatedly applying the L'Hospitol rule to (14), the following results are obtained.

1) If $k_i>k_j$, then, see the unnumbered equation at the top of this page.

Thus, the theorem follows.

2) If $k_i=k_j$, then $\lim_{t\to 0}\frac{I^c(i,t)}{I^c(j,t)}=\lim_{t\to 0}\frac{e^{\Lambda(t)}-1+F(k_i,0_i)}{e^{\Lambda(t)}-1+F(k_j,0_j))}=\frac{F(k_i,0_i)}{F(k_j,0_j))}$ since $F(k_i,0_i)>F(k_j,0_j)$, then $\lim_{t\to 0}\frac{I^c(i,t)}{I^c(j,t)}>1$. Thus, the theorem follows.

Proof of Theorem 6: Based on the fact $F(1)\le F(2)\le\cdots\le F(n)=1, F(k,0_i)\le F(k)$, and by the definition of *a*, it can be seen that $F(k,0_i)=F(k)=0$, and $\overline{F}(k)=1-F(k)=1$ for all *k* < *a*.

Thus, from (10), we have

$$I^c(i,t)=\frac{\sum_{k=a}^{n-1}\frac{\Lambda(t)^k}{k!}F(k,0_i)+e^{\Lambda(t)}-\sum_{k=0}^{n-1}\frac{\Lambda(t)^k}{k!}}{e^{\Lambda(t)}-\sum_{k=0}^{a-1}\frac{\Lambda(t)^k}{k!}-\sum_{k=a}^{n-1}\frac{\Lambda(t)^k}{k!}\overline{F}(k)}.$$

Due to (13), it can be concluded that

$$\lim_{t\to 0}I^c(i,t)=\lim_{t\to 0}\frac{e^{\Lambda(t)}-1-\Lambda(t)-\frac{\Lambda(t)^2}{2!}-\cdots-\frac{\Lambda(t)^{a-1}}{(a-1)!}-\frac{\Lambda(t)^a}{a!}(1-F(a,0_i))}{e^{\Lambda(t)}-1-\Lambda(t)-\frac{\Lambda(t)^2}{2!}-\cdots-\frac{\Lambda(t)^{a-1}}{(a-1)!}-\frac{\Lambda(t)^a}{a!}\overline{F}(a)}=\frac{F(a,0_i)}{1-\overline{F}(a)}=\frac{F(a,0_i)}{F(a)}$$

where the last equality follows from the repeated use of the L'Hospitol rule.

Since $F(a,0_i)>F(a,0_j)$, then $\lim_{t\to 0}(I^c(i,t)-I^c(j,t))=\frac{F(a,0_i)-F(a,0_j)}{F(a)}>0$. Hence, the theorem follows.

Proof of Theorem 7: The proof is similar to Theorem 4, and is therefore omitted.

Proof of Theorem 8: Similar to Theorem 5 and the proof is therefore omitted.

Proof of Theorem 9: Similar to Theorem 6 and the proof is therefore omitted.

Acknowledgment

The authors sincerely thank the anonymous reviewers for constructive comments on this paper.

References

[1] Z. W. Birnbaum, "On the importance of different components in a multicomponent system," in *Multivariate Analysis II*, P. R. Krishnaiah, Ed. New York, NY, USA: Academic, 1969, pp. 581–592.
[2] N. D. Singpurwalla, *Reliability and Risk: A Bayesian Perspective (Wiley Series in Probability and Statistics)*. New York, NY, USA: Wiley, 2006.
[3] W. Kuo and M. J. Zuo, *Optimal Reliability Modeling: Principles and Applications*. New York, NY, USA: Wiley, 2003.
[4] K. Shen and M. Xie, "On the increase of system reliability by parallel redundancy," *IEEE Trans. Rel.*, vol. 39, no. 5, pp. 607–611, Dec. 1990.
[5] P. J. Boland, E. El-Neweihi, and F. Proschan, "Active redundancy allocation in coherent systems," *Probab. Eng. Inf. Sci.*, vol. 2, no. 3, pp. 343–353, 1988.
[6] W. E. Vesely, "A time-dependent methodology for fault tree evaluation," *Nucl. Eng. Des.*, vol. 13, no. 2, pp. 337–360, 1970.
[7] J. B. Fussell, "How to hand-calculate system reliability and safety characteristics," *IEEE Trans. Rel.*, vol. R-24, no. 3, pp. 169–174, Aug. 1975.
[8] R. E. Barlow and F. Proschan, "Importance of system components and fault tree events," *Stoch. Process. Appl.*, vol. 3, no. 2, pp. 153–173, 1972.
[9] S. B. Si, H. Y. Dui, Z. Q. Cai, S. D. Sun, and Y. F. Zhang, "Joint integrated importance measure for multi-state transition systems," *Commun. Statist., Theory Methods*, vol. 41, no. 21, pp. 3846–3862, 2012.
[10] M. J. Armstrong, "Joint reliability-importance of components," *IEEE Trans. Rel.*, vol. 44, no. 3, pp. 408–412, Sep. 1995.
[11] S. M. Wu, "Joint importance of multistate systems," *Comput. Ind. Eng.*, vol. 49, no. 1, pp. 63–75, 2005.
[12] X. Gao, L. Cui, and J. Li, "Analysis for joint importance of components in a coherent system," *Eur. J. Oper. Res.*, vol. 182, no. 1, pp. 282–299, 2007.
[13] F. K. Hwang, "A hierarchy of importance indices," *IEEE Trans. Rel.*, vol. 54, no. 1, pp. 169–172, Mar. 2005.
[14] F. C. Meng, "Comparing the importance of system components by some structural characteristics," *IEEE Trans. Rel.*, vol. 45, no. 1, pp. 59–65, Mar. 1996.
[15] P. J. Boland, F. Proschan, and Y. L. Tong, "Optimal arrangement of components via pairwise rearrangements," *Nav. Res. Logist.*, vol. 36, no. 6, pp. 807–815, 1989.
[16] J. Freixas and M. Pons, "Identifying optimal components in a reliability system," *IEEE Trans. Rel.*, vol. 57, no. 1, pp. 163–170, Mar. 2008.
[17] F. K. Hwang, "A new index of component importance," *Oper. Res. Lett.*, vol. 28, no. 2, pp. 75–79, 2001.
[18] D. Butler, "Complete importance ranking for components of binary coherent systems with extensions to multi-state systems," *Nav. Res. Logist. Quart.*, vol. 26, no. 4, pp. 565–578, 1979.
[19] L. B. Page and J. E. Perry, "Reliability polynomials and link importance in networks," *IEEE Trans. Rel.*, vol. 43, no. 1, pp. 51–58, Mar. 1994.

[20] J. S. Hong and C. H. Lie, "Joint reliability-importance of two edges in an undirected network," *IEEE Trans. Rel.*, vol. 42, no. 1, pp. 17–23, Mar. 1993.
[21] I. B. Gertsbakh and Y. Shpungin, "Combinatorial approach to computing component importance indexes in coherent systems," *Probab. Eng. Inf. Sci.*, vol. 26, no. 1, pp. 117–128, 2012.
[22] I. B. Gertsbakh and Y. Shpungin, *Models of Network Reliability: Analysis, Combinatorics, and Monte Carlo*. Boca Raton, FL, USA: CRC Press, 2009.
[23] R. Vaisman, D. P. Kroese, and I. B. Gertsbakh, "Improved sampling plans for combinatorial invariants of coherent systems," *IEEE Trans. Rel.*, vol. 65, no. 1, pp. 410–424, Mar. 2016.
[24] I. B. Gertsbakh and Y. Shpungin, *Network Reliability and Resilience*. Berlin, Germany: Springer-Verlag, 2011.
[25] S. Zarezadeh and M. Asadi, "Network reliability modeling under stochastic process of component failures," *IEEE Trans. Rel.*, vol. 62, no. 4, pp. 917–929, Dec. 2013.
[26] S. Zarezadeh, S. Ashrafi, and M. Asadi, "A shock model based approach to network reliability," *IEEE Trans. Rel.*, vol. 65, no. 2, pp. 992–1000, Jun. 2016.
[27] T. Nakagawa, *Stochastic Processes With Applications to Reliability Theory*. London, U.K.: Springer, 2011.
[28] S. Zarezadeh, M. Asadi, and N. Balakrishnan, "Dynamic network reliability modeling under nonhomogeneous Poisson processes," *Eur. J. Oper. Res.*, vol. 232, no. 3, pp. 561–571, 2014.
[29] H. W. Chang and F. K. Hwang, "Rare-event component importance for the consecutive-k system," *Nav. Res. Logist.*, vol. 49, no. 2, pp. 159–166, 2002.

Yongjun Du received the M.S. degree in applied mathematics in 2007 from Northwestern Polytechnical University, Xi'an, China, where he is working toward the Ph.D. degree at the Department of Industrial Engineering, School of Mechanical Engineering.

He is also a Lecturer with the Department of Applied Mathematics, School of Science, Lanzhou University of Technology, Lanzhou, China. His research interests include network reliability and importance analysis.

Shubin Si (M'09–SM'16) received the B.S. and M.S. degrees in mechanical engineering and the Ph.D. degree in management science and engineering from Northwestern Polytechnical University (NPU), Xi'an, China, in 1997, 2002, and 2006, respectively.

He is a Professor with the School of Mechanical Engineering, NPU. He has published more than 60 academic papers and articles in journals and conferences in the past 5 years. He also headed and participated in five government supported foundations and more than ten enterprise supported projects. His research interests include system reliability theory, importance measures, and maintenance management systems.

Dr. Si is a Senior Member of the IEEE Reliability Society.

Tongdan Jin (M'07) received the Ph.D. degree in industrial and systems engineering from Rutgers University, New Brunswick, NJ, USA, in 2001.

He is an Associate Professor with the Ingram School of Engineering, Texas State University, San Marcos, TX, USA. Prior to academia, he has five years of reliability management experience in Teradyne. Inc., Boston, MA, USA. His current research interests focus on system reliability modeling and optimization, product-service integration, and low-carbon supply chain operations.

Dr. Jin is the recipient of Evans/McElroy best paper in the 2014 Reliability and Maintainability Symposium. He is a member of the IEEE Reliability Society.