Optimal Control of a Twin Rotor MIMO System Using LQR with Integral Action

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***Abstract* - The twin rotor MIMO system (TRMS) is a helicopter-like system that is restricted to two degrees of freedom, pitch and yaw. It is a complicated nonlinear, coupled, MIMO system used for system identification, the verification of control methods and observers. This paper details the design procedure for a suboptimal tracking controller using a *linear quadratic regulator* (LQR) with integral action. It was found that the *LQR controller with integral action* (LQI) provided performance superior to existing optimal controllers in the literature.**

I. Introduction

The TRMS is a common control problem for validating new control methodologies. The simplest solution to the problem is to apply PID control to the system. There has also been a lot of research into applying machine learning to the system to find the optimal parameters to guarantee the best performance [1]. However these machine learning algorithms do not ensure robust performance, which is desired in aerospace applications. To ensure robust performance various control methods such as deadbeat control [2], optimal control [3] [4] [5], H_∞ [6] and sliding mode control [7] [8] [9] [10] has been applied to the system as well.

The simplest solution to most any control problem is to use a hand tuned PID controller, the TRMS is no exception. However in an effort to take the human out of the slow tuning process the PID controller is typically tuned via a machine learning process. In 2012 Meon M.S. et al proposed a method called PID Active Force Control [1] in which a system estimated the external torque disturbances and a neural network and fuzzy logic were used to optimize the PID controller. The method provided a smooth, albeit slow, response that worked well at rejecting disturbances. However the controller was only implemented in simulation and the results for the yaw subsystem were never reported [1].

A more complex option is to use a Linear Gaussian based controller which has the benefit of having its own robustness properties [11] [12] [13]. There are two forms of the LG controllers that have been implemented in simulation. The first is the LQR design, which requires a linear system, the TRMS is a nonlinear system so it must be put into a linear form. It also requires full state feedback so a suitably robust observer must be implemented. The second form is the *Linear Quadratic Gaussian* (LQG) controller which combines the LQR controller with a Kalman filter. The advantage of this method is that an optimal estimator is used to provide the full state feedback. Again the system must be put into a linear form to utilize this method.

In 2004 A.Q. Khan used the LQR method to control a 3 DOF helicopter [14]. The steady state solution was applied to the linearized system model to implement full state feedback. The controller provided adequate performance around the equilibrium point. However the controller was not considered robust in the face of uncertainties and future work was needed to make it so. In 2012 B. Pratap demonstrated a new approach to the LQR problem applied to the TRMS in simulation. The system was decoupled into two subsystems and then linearized [5] an approach that was made use of in other implementations [9] [10] [8] [7]. Two LQR controllers were designed for the subsystems, however instead of taking the suboptimal solution to the LQR problem the Kalman gain was updated iteratively until the cost function was minimized. The controller was used simply as a regulating controller though, so no conclusion could be drawn on the tracking results [5].

As stated before another method is to use LQG controllers. In 2000 S.M. Ahmed et al applied the LQG method to a one degree of freedom (DOF) helicopter [4]. The LQG compensator failed to provide adequate performance alone so a prefilter was used to attenuate high frequency vibration and lower the command effort required. Another implementation of the LQG controller was demonstrated in 2013 by A.K. Agrawal. It showed through simulation that a LQG controller with weighting matrices optimized using a bacterial foraging method to control the TRMS [3]. The resulting controller showed improved performance over the controller designed using manually weighted matrices.

In this paper a suboptimal controller solution with integral action is proposed, designed, simulated and implemented experimentally. In part two the TRMS system of equations are derived. In part three the LQR control theory is derived, the method for imparting integral action on the loop is presented, and the robustness properties are examined. In part four the controller is designed by linearizing the plant about the equilibrium point. In part five the simulation parameters are stated and the results are presented. In part six the experimental parameters are stated and the experimental results are presented. In part seven conclusions are drawn and future work is proposed.

II. The Twin Rotor MIMO System

A. *Derivation of Plant Model*

The TRMS is a laboratory setup provided by Feedback Instruments for the purpose of testing new controllers. The TRMS is characterized by highly coupled, non-linear dynamics. The setup consists of a horizontal beam fixed to a vertical pillar via a two dimensional pivot. The main rotor is affixed to the front of the horizontal beam parallel to the ground. The tail rotor is affixed to the rear of the horizontal beam perpendicular to the ground. A counterbalance beam is affixed to the horizontal beam at the pivot to move the equilibrium point of the system. The main and tail rotors are controlled by two DC motors.

The DC motors are controlled using a DAC card that is installed in a desktop PC. The DAC card is sent commands from the controller that is designed using MATLAB and SIMULINK. Real time control of the TRMS from MATLAB is possible using the real time windows target.

The plant used in the controller design is from [10]. The plant constants can be seen in Table 1. The state variables are defined as Equation (1). The system of equations for the plant can be written as Equation (2).

Table 1: Parameter Definitions of the TRMS.

Parameter	***Description***	***Value***	***Units***
a_1	Main Rotor Coefficient	.0135	N/A
b_1	Main Rotor Coefficient	.0924	m
a_2	Tail Rotor Coefficient	.01	m
b_2	Tail Rotor Coefficient	.09	m
$B_{1\theta_V}$	Friction Momentum	.003	Nm s/rad
$B_{1\theta_H}$	Friction Momentum	.1	Nm s/rad
M_g	Moment of Gravity	.29	Nm
I_1	Pitch Moment of Inertia	.0535	Kg m^2
I_2	Yaw Moment of Inertia	.02	Kg m^2
K_{gy}	Gyroscopic Momentum	.05	s/rad
T_P	Cross Reaction Momentum Parameter	2	N/A
T_0	Cross Reaction Momentum Parameter	3.5	N/A
K_C	Cross Reaction Momentum Gain	-.2	N/A
T_{10}	Main Rotor Denominator	1	N/A
T_{11}	Main Rotor Denominator	1.1	N/A
T_{20}	Tail Rotor Denominator	1	N/A
T_{21}	Tail Rotor Denominator	1	N/A

$$\boldsymbol{X} = \begin{bmatrix} \theta_V \\ \Omega_V \\ \theta_H \\ \Omega_H \\ M_R \\ \tau_1 \\ \tau_2 \end{bmatrix} \tag{1}$$

$$\begin{gathered}
\frac{d\theta_V}{dt} = \Omega_V \\
\frac{d\Omega_V}{dt} = \frac{1}{I_1}\Big[a_1\tau_1^2 + b_1\tau_1 - M_g\sin(\theta_V) - B_{1\theta_V}\Omega_V + \frac{.0326}{2}\sin(2\theta_V)\,\Omega_V^2 - K_{gy}a_1\cos(\theta_V)\,\Omega_H\tau_1^2 - \\
K_{gy}b_1\cos(\theta_V)\,\Omega_H\tau_1\Big] \\
\frac{d\theta_H}{dt} = \Omega_H \\
\frac{d\Omega_H}{dt} = \frac{1}{I_2}\left[a_2\tau_2^2 + b_2\tau_2 - B_{1\theta_H}\Omega_H - \left(\frac{K_C}{T_P} - \frac{K_C T_P}{T_P^2}\right)M_R - \frac{K_C T_0}{T_P}(a_1\tau_1^2 + b_1\tau_1)\right] \\
\dot{M}_R = -\frac{1}{T_P}M_R + a_1\tau_1^2 + b_1\tau_1 \\
\frac{d\tau_1}{dt} = -\frac{T_{10}}{T_{11}}\tau_1 + \frac{K_1}{T_{11}}U_V \\
\frac{d\tau_2}{dt} = -\frac{T_{20}}{T_{21}}\tau_2 + \frac{K_2}{T_{21}}U_H
\end{gathered} \tag{2}$$

III. Optimal Control

A. LQR Control

Designing a controller for a system by placing the poles results in a system of equations that is *overdetermined*; that is there are more equations than variables to be solved for. Because there is more than one solution to this problem there must be one that is quantifiably better than the others. This gave birth to the concept of Optimal Control Theory. There are many solutions to the optimal control theory such as LQR, LQG, and dynamic programming.

The derivation of the LQR controller is well known [15]. Consider a continuous linear system shown in Equation (3).

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} \tag{3}$$

The optimal input is given by Equation (4). Here $\boldsymbol{K}$ is called "The Kalman Gain".

$$\boldsymbol{u} = -\boldsymbol{R}^{-1}\boldsymbol{B}^{T}\boldsymbol{S}\boldsymbol{x} = -\boldsymbol{K}\boldsymbol{x} \tag{4}$$

$\boldsymbol{S}$ is given by the Matrix Riccati Equation in Equation (5).

$$-\dot{\boldsymbol{S}} = \boldsymbol{A}^{T}\boldsymbol{S} + \boldsymbol{S}\boldsymbol{A} - \boldsymbol{S}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{T}\boldsymbol{S} + \boldsymbol{Q} \tag{5}$$

Equation (5) is time varying so it provides a set of differential equations that must be solved in conjunction with the required control effort. However this is not always be the case. If a steady state solution to Equation (5) is found then that can be used instead of the time varying solution. This is called the *suboptimal solution.*

B. Suboptimal Control

It can be seen from Equations (4) and (5) that the Kalman Gain is time varying. However the optimal Kalman Gain matrix can be approximated in what is called "the suboptimal solution" [15]. This is done by assuming a steady state solution to Equation (5). This is shown in Equation (6).

$$\boldsymbol{A}^{T}\boldsymbol{S} + \boldsymbol{S}\boldsymbol{A} - \boldsymbol{S}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{T}\boldsymbol{S} + \boldsymbol{Q} = \boldsymbol{0} \tag{6}$$

So now the Kalman gain is no longer time varying. The only requirement is that the matrix $\boldsymbol{S}$ satisfies Equation (6).

By implementing the controller using the above equations to design can be relatively simple. The difficulty in the design is due to two reasons. The first is due to trying to properly weight the $\boldsymbol{Q}$ and $\boldsymbol{R}$ matrices to meet the performance requirements. The second is that integral action is not imparted on the loop by default. This can result in bad or even no tracking properties. However there is a method that can be used to guarantee that integral action is imparted on the loop. This is described in the next section.

C. Adding Integral Action

In many situations it is desired that the control loop has integral action to guarantee no steady state error to a step input. To impart integral action on the loop, the system is restated in a form that creates a number of additional states equal to the number of outputs that are the output error of the system.

By augmenting the state space system as shown in Equations (7) and (8), integral action will be imparted onto the loop [16].

$$\widehat{\boldsymbol{A}} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} \tag{7}$$

$$\widehat{\boldsymbol{B}} = \begin{bmatrix} B \\ 0 \end{bmatrix} \tag{8}$$

The A matrix must remain square, so from inspection it can be seen that the effect of this augmentation is the addition of a number of poles at the origin equal to the number of outputs. The suboptimal solution to the augmented system can be solved the same way as detailed in part 3B.

In addition to integral action another characteristic that is desired of a controller is robustness. Robustness allows a system to continue to function properly in the face of changes of system parameters or dynamics. There are four types of control theory that can guarantee robustness: deadbeat, robust control theory, sliding mode control and Linear Gaussian based controllers.

D. Robustness Properties

It is well known that the Linear Gaussian controllers exhibit robust properties [13] [12]. It can be shown that LQR controllers can tolerate gain attenuation of between ½ and ∞ and a phase disturbance between ±60° by defining a quadratic Lyapunov candidate as a function of Equation (5).

However these robustness properties are not always met. It has been shown by [13] that it is possible to choose $\boldsymbol{Q}$ and $\boldsymbol{R}$ values such that these conditions are not met [13]. For this reason the robustness of the loop must be analyzed to determine the stability of the closed loop system.

IV. LQR Control Design

A. Plant Linearization

The LQR solution presented here requires a linear system. The nonlinear system presented in Equation (2) is linearized using Equations (8) and (9).

$$A = \left.\frac{dF}{dx}\right|_0 \tag{8}$$

$$B = \left.\frac{dF}{du}\right|_0 \tag{9}$$

Applying Equations (8) and (9) to the Equation (2), Equations (10) and (11) are found. These are the linear model of the plant.

$$\boldsymbol{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{10}$$

$$\boldsymbol{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{K_1}{T_{11}} & 0 \\ 0 & \frac{K_2}{T_{21}} \end{bmatrix} \tag{11}$$

In Equation (10) the A matrix is represented in block form for readability. The A_{11}, A_{12}, A_{21}, and A_{22} matrices are shown in Equations (12) through (15).

$$\boldsymbol{A_{11}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{M_g}{I_1} & -\frac{B_{1\theta_V}}{I_1} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\frac{B_{1\theta_H}}{I_2} \end{bmatrix} \tag{12}$$

$$\boldsymbol{A_{12}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{2a_1+b_1}{I_1} & 0 \\ 0 & 0 & 0 \\ \frac{K_C(T_0+T_P)}{T_P^2} & \frac{K_C T_0(2a_1+b_1)}{I_2 T_P} & \frac{2a_2+b_2}{I_2} \end{bmatrix} \tag{13}$$

$$\boldsymbol{A_{21}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{14}$$

$$\boldsymbol{A_{22}} = \begin{bmatrix} -\frac{1}{T_P} & 2a_1 + b_1 & 0 \\ 0 & -\frac{T_{10}}{T_{11}} & 0 \\ 0 & 0 & -\frac{T_{20}}{T_{21}} \end{bmatrix} \tag{15}$$

It can be seen that there is already a pole at the origin. However this pole is a result of the linearization, and the location of the poles will change with the system state. The guarantee that there will be zero steady state error in a step input integral action will be imparted onto the suboptimal loop.

V. Simulation Results

The LQI controller was implemented in hardware using the Twin Rotor MIMO system supplied by Feedback Instruments. The simulations were done using MATLAB and SIMULINK and a sample time of "1ms" and the "ode5" solver was used. A Luenberger observer was used to provide the full state feedback. A PID controller was implemented to provide a point of comparison.

The Q and R matrices needed for the LQI controller design are given by Equations (16) and (17).

$$\boldsymbol{Q} = diag([10^4; 10^4; 10^3; 10^3; 1; 1; 1; 10^4; 10^4]) \tag{16}$$

$$\boldsymbol{R} = \boldsymbol{I} \tag{17}$$

A. Comparison of Simulation Results

The LQI controller was compared to the results obtained by Ankesh Kumar Agrawal [3] in simulation. Figure 1 shows the results obtained from A.K. Agrawal using bacterial forging to optimize the $\boldsymbol{Q}$ and $\boldsymbol{R}$ matrices of a LQG controller. Figure 2 shows the results of the LQI controller in simulation for the same desired response. It is easy to see that the LQI controller provided superior response to the results obtained by A.K. Agrawal due to a decreased rise and settling time.

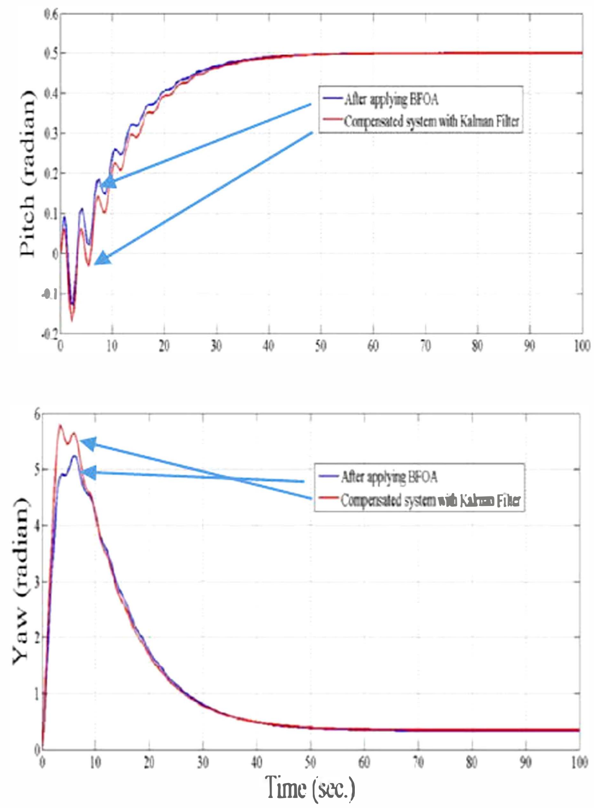

Figure 1: Simulated pitch and yaw step by A.K. Agrawal [3].

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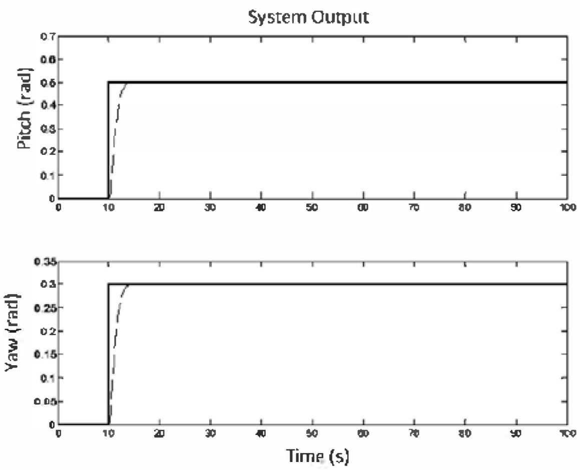

Figure 2: Simulated pitch and yaw step replicating results from [3].

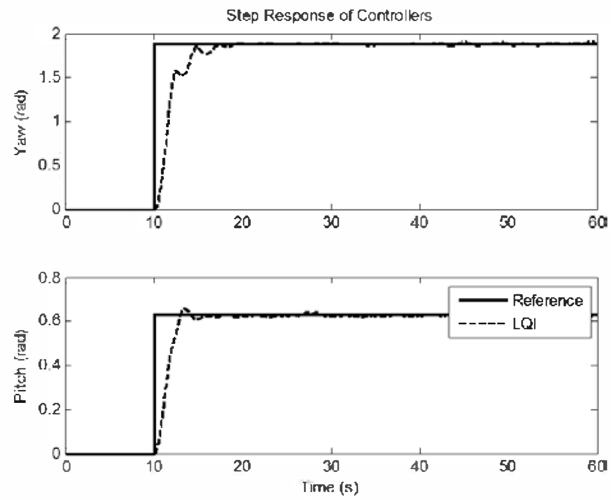

Figure 3: Experimental step response of the TRMS.

VI. Experimental Results

In addition to being implemented in simulation the LQI controller was also implemented experimentally using the Twin Rotor MIMO system supplied by Feedback Instruments. As with the simulation, a Luenberger Observer was used to provide full state feedback. The $\boldsymbol{Q}$ and $\boldsymbol{R}$ matrices for the experimental LQI controller are given by Equations (18) and (19).

$$\boldsymbol{Q} = diag([10^3; 200; 10; 10; 1; 1; 1; 10^3; 100]) \quad (18)$$

$$\boldsymbol{R} = \boldsymbol{I} \quad (19)$$

A. Comparison of Hardware Results

There has been no experimental results published on the implementation of a LQR or LQG controller applied to a twin rotor system. Because of this the LQI controller is compared experimentally against another form of robust control, sliding mode control.

In 2013 D.K. Saroj et al [10] implemented a sliding mode controller with a nonlinear state observer based off of the Luenberger structure experimentally. The reference signals for the LQI controller were chosen to replicate the results from D.K. Saroj et al.

The experimental results were reported in the form of a step response. Figure 3 shows the step response of the LQI controller and Figure 4 shows the step response of the sliding mode controller.

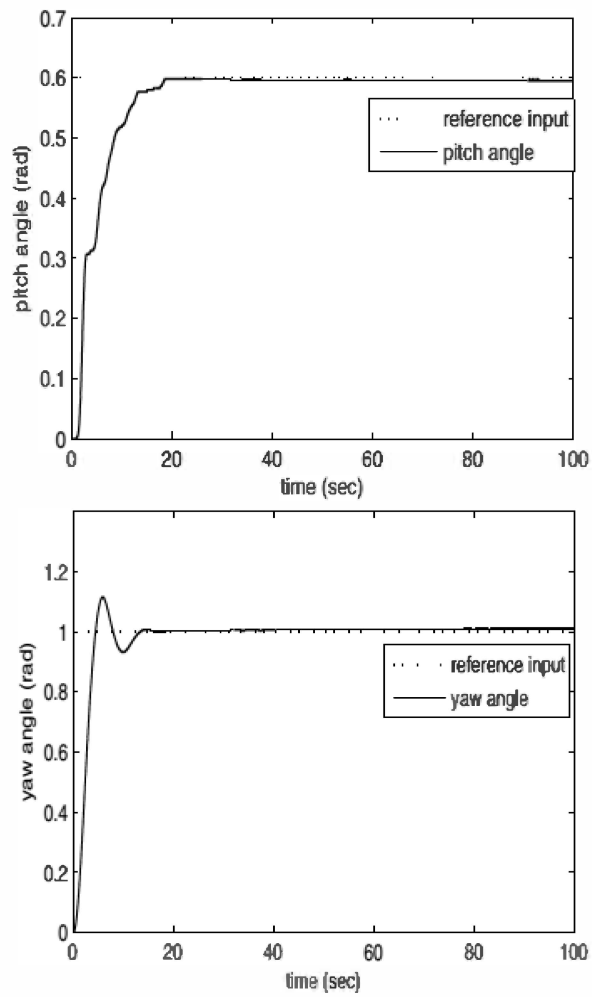

Figure 4: Experimental pitch and yaw step from D.K. Saroj et al [10].

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In Figure 4 it is seen that in the pitch subsystem has a settling time of about 20 seconds with no overshoot. The yaw subsystem has a settling time of about 15 seconds. In Figure 3 the pitch subsystem has a settling time of about 5 seconds and the yaw subsystem has a settling time of about 7 seconds with no overshoot. Because of the faster rise time the LQI controller provides better performance than the sliding mode controller.

VII. Conclusions

An LQI controller was designed, simulated, implemented experimentally and compared against existing optimal and robust controllers in the literature. It was found that the LQI controller proposed here provided superior performance to the existing controller solutions.

In simulation it was shown that the LQI controller provided better performance than the LQG bacterial forging algorithm [3]. Experimentally it was found that the LQI controller provided better step response than the sliding mode controller implemented by [10].

An area of future work is to implement a linear or nonlinear optimal tracking controller. This would guarantee perfect tracking of all system states while maintaining the robustness characteristics mentioned above.

VIII. Works Cited

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