Modern Physics Letters B
Vol. 28, No. 3 (2014) 1450021 (9 pages)

DOI: 10.1142/S0217984914500213

WAVE PROPAGATION IN RECTANGULAR NANOPLATES BASED ON STRAIN GRADIENT THEORY WITH ONE GRADIENT PARAMETER WITH CONSIDERING INITIAL STRESS

MOHAMMAD RAHIM NAMI and MAZIAR JANGHORBAN*
School of Mechanical Engineering, Shiraz University, Shiraz, Iran
**maziar.janghorban@gmail.com*

Received 21 November 2013
Revised 2 December 2013
Accepted 3 December 2013
Published 7 January 2014

In this paper, on the basis of gradient elasticity theory with one gradient parameter, wave propagation in rectangular nanoplates is studied. In the governing equation, the influences of initial stresses and elastic foundation are also considered. An analytical approach is used to solve the governing equation. The effects of different parameters such as gradient parameter on the circular and cut-off frequencies are presented. One can see that the initial stress and gradient parameter play an important role in investigating the wave propagation in nanoplates.

Keywords: Strain gradient elasticity theory; wave propagation; rectangular nanoplates; initial stress; elastic matrix.

1. Introduction

Wave propagation seems to have different application in macro- and nanostructures such as vibration control, damage detection, computation of elastic constants, aviation and transportation. In recent years, a number of studies of wave propagation in nanostructures have been carried out through theoretical modeling and computer simulation.[1] Narendar and Gopalakrishnan[2] investigated the thermal effects on the ultrasonic wave propagation characteristics of a nanoplate based on the nonlocal continuum theory. The axial stress caused by the thermal effects was considered. The wave propagation analysis was carried out using spectral analysis. Wang *et al.*[3] studied the small-scale effects on the flexural wave in the nanoplate. Based on the nonlocal continuum theory, the equation of wave motion was derived and the

*Corresponding author.

dispersion relation was presented with considering the initial stresses and elastic matrix. Arash *et al.*[4] developed a nonlocal elastic plate model that accounts for the scale effects for wave propagations in graphene sheets. Moreover, a finite element model developed from the weak-form of the elastic plate model was reported to fulfill a comprehensive wave study in the sheets. Based on the Bernoulli–Euler and Timoshenko beam theories, a single-elastic beam model using nonlocal elasticity was developed for the wave propagation in carbon nanotubes (CNTs) by Heireche *et al.*[5] Frequency equations and modal shape functions of Timoshenko beams structures with some typical boundary conditions were also derived from nonlocal elasticity. Wang *et al.*[6] proposed the propagation characteristics of the longitudinal wave in nanoplates with small scale effects. The equation of the longitudinal wave was obtained using the nonlocal elastic theory. The phase velocity and the group velocity were also derived, respectively. Liu and Yang[7] presented the propagation of elastic waves in a single-layered graphene sheet supported by an elastic medium via the nonlocal continuum model. The elastic medium was treated as a two-parameter elastic foundation. The governing equations accounting for coupled longitudinal and vertically polarized shear waves were obtained and dispersion relations were given. Besseghier *et al.*[8] studied the thermal effect on wave propagation in double-walled carbon nanotubes (DWNTs) embedded in a polymer matrix via nonlocal elasticity. The small-scale effects on vibration characteristics of CNTs were explicitly derived through a complete continuum beam model. Narendar and Gopalakrishnan[9] investigated the effect of nonlocal scale parameter on the wave propagation in multi-walled carbon nanotubes (MWCNTs). Each wall of the MWCNT was modeled as first order shear deformation beams and the van der Waals interactions between the walls were modeled as distributed springs. The nonlocal elasticity theory had been incorporated into classical Euler–Bernoulli rod model to capture unique features of the nanorods under the umbrella of continuum mechanics theory by Narendar and Gopalakrishnan.[10] The analysis showed that the wave characteristics are highly over estimated by the classical rod model, which ignores the effect of small-length scale. Narendar and Gopalakrishnan[11] also studied the strong nonlocal scale effect on the flexural wave propagation in a monolayer graphene sheet. The graphene was modeled as an isotropic plate of one atom thick. Nonlocal governing equation of motion was derived and wave propagation analysis was performed using spectral analysis. Assadi and Farshi[12] proposed the size-dependent free vibration of nanotubes with surface effects. An efficient shell-core-shell model was introduced to simulate the structure which includes the effect of additional surface elasticity. Love's continuum model for longitudinal wave propagation was employed, which accounts for the effects of lateral contractions.

In this work, the wave propagation in rectangular nanoplates based on Kirchhoff plate theory using gradient elasticity theory with one gradient parameter is presented. The effects of initial stress and elastic matrix on the circular and cut-off frequencies are also proposed. In this study, the Winkler foundation and the shearing layer are considered to model the elastic matrix. Our numerical results are also verified with the results of wave propagation in macro plates. To the best of the authors' knowledge, it is for the first time that strain gradient elasticity theory is used to investigate the wave propagation in rectangular nanoplates.

2. Strain Gradient Theory

The strain/stress greatly enhances the electronic, structural, magnetic, and optical properties of the systems.[13–18] For a linear isotropic elastic material, stresses are explained by the kinematic parameters effective on the strain density which are given in the following constitutive relations,[19]

$$\begin{aligned} \sigma_{ij} &= \lambda\delta_{ij}\varepsilon_{mm} + 2\mu\varepsilon_{ij}\,, \\ P_i &= 2\mu l_0^2\gamma_i\,, \\ \tau_{ijk}^{(1)} &= 2\mu l_1^2\eta_{ijk}^{(1)}\,, \\ m_{ij}^s &= 2\mu l_2^2 X_{ij}^s\,, \end{aligned} \tag{1}$$

where ε_{ij}, $\gamma_i, \eta_{ijk}^{(1)}$, X_{ij}^s, and l_i are the strain, dilatation gradient, deviatoric stretch gradient, symmetric rotation gradient tensors and material length scale parameters. P_i, $\tau_{ijk}^{(1)}$, m_{ij}^s are the higher-order stresses, The simplest form of above relations can be expressed as follows:[20]

$$(\sigma_{ij}) = C_{ijkl}(\varepsilon_{ij} - l\varepsilon_{ij,mm})\,, \tag{2}$$

where C_{ijkl} and l are the elastic constants and gradient parameter, respectively. The values of gradient constant can be found in Papargyri–Beskou and Beskos.[20] Above equation will be used to consider the size effects in studying wave propagation in rectangular nanoplates.

3. Governing Equations

In this section, in order to study the wave propagation in rectangular nanoplates, the gradient elasticity theory with one gradient parameter and Kirchhoff plate theory are used to derive the governing equation of motion. The above constitutive relations (2) can be expressed in long form as below.

Equation (2):

$$\sigma_x = \frac{E}{1-v^2}(\varepsilon_x + v\varepsilon_y) - \frac{E}{1-v^2} l\nabla^2(\varepsilon_x + v\varepsilon_y)\,. \tag{3}$$

Equation (2):

$$\sigma_y = \frac{E}{1-v^2}(\varepsilon_y + v\varepsilon_x) - \frac{E}{1-v^2} l\nabla^2(\varepsilon_y + v\varepsilon_x)\,. \tag{4}$$

Equation (2):

$$\tau_{xy} = \frac{E}{2(1+v)}(\gamma_{xy}) - \frac{E}{2(1+v)} l\nabla^2(\gamma_{xy})\,, \tag{5}$$

where E is the Young modulus and v is the Poisson's ratio. To have the stresses in terms of displacements, following relations between strain and displacements should be used:

$$\varepsilon_x = -z\frac{\partial^2 w}{\partial x^2}, \quad \varepsilon_y = -z\frac{\partial^2 w}{\partial y^2}, \quad \gamma_{xy} = -2z\frac{\partial^2 w}{\partial x \partial y}\,. \tag{6}$$

Now by substituting the above equation in Eqs. (3)–(5), the stress–displacement relations with considering gradient parameter can be written as follows:

Equation (3):

$$\begin{aligned} \sigma_x = {} & \frac{E}{1-v^2}\left(-z\frac{\partial^2 w}{\partial x^2} - vz\frac{\partial^2 w}{\partial y^2}\right) \\ & - \frac{E}{1-v^2}l\left(-z\frac{\partial^4 w}{\partial x^4} - z\frac{\partial^4 w}{\partial x^2 \partial y^2} - vz\frac{\partial^4 w}{\partial y^4} - vz\frac{\partial^4 w}{\partial x^2 \partial y^2}\right). \end{aligned} \tag{7}$$

Equation (4):

$$\begin{aligned} \sigma_y = {} & \frac{E}{1-v^2}\left(-z\frac{\partial^2 w}{\partial y^2} - vz\frac{\partial^2 w}{\partial x^2}\right) \\ & - \frac{E}{1-v^2}l\left(-z\frac{\partial^4 w}{\partial x^4} - z\frac{\partial^4 w}{\partial x^2 \partial y^2} - vz\frac{\partial^4 w}{\partial x^4} - vz\frac{\partial^4 w}{\partial x^2 \partial y^2}\right). \end{aligned} \tag{8}$$

Equation (5):

$$\tau_{xy} = \frac{E}{2(1+v)}\left(-2z\frac{\partial^2 w}{\partial x \partial y}\right) - \frac{E}{2(1+v)}l\left(-2z\frac{\partial^4 w}{\partial x^3 \partial y} - 2z\frac{\partial^4 w}{\partial y^3 \partial x}\right). \tag{9}$$

The next step is to construct the gradient resultant moments in term of displacements by using Eqs. (7)–(9) as follows:

Equation (7):

$$M_x = D\left(-\frac{\partial^2 w}{\partial x^2} - v\frac{\partial^2 w}{\partial y^2}\right) - lD\left(-\frac{\partial^4 w}{\partial x^4} - \frac{\partial^4 w}{\partial x^2 \partial y^2} - v\frac{\partial^4 w}{\partial y^4} - v\frac{\partial^4 w}{\partial x^2 \partial y^2}\right). \tag{10}$$

Equation (8):

$$M_y = D\left(-\frac{\partial^2 w}{\partial y^2} - v\frac{\partial^2 w}{\partial x^2}\right) - lD\left(-\frac{\partial^4 w}{\partial y^4} - \frac{\partial^4 w}{\partial y^2 \partial x^2} - v\frac{\partial^4 w}{\partial x^4} - v\frac{\partial^4 w}{\partial y^2 \partial x^2}\right). \tag{11}$$

Equation (9):

$$M_{xy} = D(1-v)\left(2\frac{\partial^2 w}{\partial x \partial y}\right) - lD(1-v)\left(2\frac{\partial^4 w}{\partial y^3 \partial x} + 2\frac{\partial^4 w}{\partial y \partial x^3}\right), \tag{12}$$

where $(M_x, M_y, M_{xy}) = \int_{-h/2}^{h/2}(\sigma_x, \sigma_y, -\tau_{xy})z dz$ and $D = \frac{Eh^3}{12(1-v^2)}$. The dynamic equilibrium equation of a nanoplate in terms of the moment resultants is given as

$$\frac{\partial^2 M_x}{\partial x^2} - 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + N_0\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = \rho h\frac{\partial^2 w}{\partial t^2} + K_w w - G_b \nabla^2 w\,. \tag{13}$$

To clearly show the effects of gradient parameter, elastic matrix and initial stress on the propagation characteristics of elastic waves in a single-layered graphene sheet, following governing equation can be achieved from Eqs. (10)–(13)

$$\begin{aligned}&\rho h\frac{\partial^2 w}{\partial t^2} - N_0\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right) + K_w w - G_b\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)\\ &\quad = D\left(-\frac{\partial^4 w}{\partial x^4}-\frac{\partial^4 w}{\partial y^4}-2\frac{\partial^4 w}{\partial x^2\partial y^2}\right)\\ &\qquad - lD\left(-\frac{\partial^6 w}{\partial x^6}-\frac{\partial^6 w}{\partial y^6}-3\left(\frac{\partial^6 w}{\partial x^4\partial y^2}+\frac{\partial^6 w}{\partial x^2\partial y^4}\right)\right). \end{aligned} \tag{14}$$

The harmonic analytical solution for the wave propagation in the rectangular nanoplates can be obtained as[3,21]

$$w = We^{i(xK_x + yK_y - \omega t)}. \tag{15}$$

where K_x and K_y are the half wave numbers in the x- and y-direction, respectively and ω is the circular frequency. It is noted that the wave number in the following investigations is defined as $K = \sqrt{K_x^2 + K_y^2}$. Now by inserting Eq. (15) in Eq. (14), the circular frequency in terms of wave numbers in the x and y directions is expressed as, K_x^4

$$\omega = \sqrt{\frac{D\left(K_x^4+K_y^4+2K_x^2K_y^2\right)+lD\left(K_x^6+K_y^6+3\left(K_x^4K_y^2+K_x^2K_y^4\right)\right)+K_w+G_b(K_x^2+K_y^2)+N_0(K_x^2+K_y^2)}{\rho h}} \tag{16}$$

where ρ is the density and h is the thickness. The above closed form solution for circular frequency of rectangular nanoplates can show the effects of different parameters such as gradient parameter, elastic matrix and initial stress on the wave propagation. In the next section, some numerical results are extracted from the above equation for gradient nanoplates.

4. Numerical Results

In this section, the numerical results for the wave propagation in rectangular nanoplates are presented on the basis of strain gradient elasticity theory with one gradient constant. In the present study, the material properties are defined as follows:[3]

$$E = 1.06\ \text{TPa}, \quad h = 0.34\ \text{nm}, \quad v = 0.25, \quad \rho = 2250\ \text{kg/m}^3,$$

$$K_w = 1.13\times 10^{18}, \quad G_b = 2\ \text{N/m}.$$

As the first example, the effects of gradient parameter and half wave number in y-direction on the cut-off frequencies are studied in Table 1. It is noted that cut-off frequencies are defined in two different ways. Some researchers introduce cut-off frequencies by assuming the wave number to be zero[3] but other researchers define

Table 1. The effects of half wave number and gradient parameter on the cut-off frequencies.

	l						
K_y (10^9)	0.0[3]	0.0	0.5	1.0	1.5	2.0	2.5
0.00	1.2154	1.2154	1.2154	1.2154	1.2154	1.2154	1.2154
0.05	1.2205	1.2205	1.2205	1.2205	1.2205	1.2205	1.2205
0.10	1.2358	1.2358	1.2358	1.2358	1.2358	1.2358	1.2358
0.15	1.2615	1.2615	1.2615	1.2615	1.2615	1.2616	1.2616
0.20	1.2976	1.2976	1.2976	1.2977	1.2977	1.2978	1.2979
0.25	1.3441	1.3441	1.3443	1.3446	1.3448	1.3450	1.3452
0.30	1.4013	1.4013	1.4019	1.4026	1.4032	1.4038	1.4044
0.35	1.4692	1.4692	1.4707	1.4722	1.4737	1.4753	1.4768
0.40	1.5480	1.5480	1.5512	1.5544	1.5576	1.5607	1.5639
0.45	1.6377	1.6377	1.6438	1.6499	1.6560	1.6621	1.6681
0.50	1.7384	1.7384	1.7493	1.7601	1.7708	1.7814	1.7920

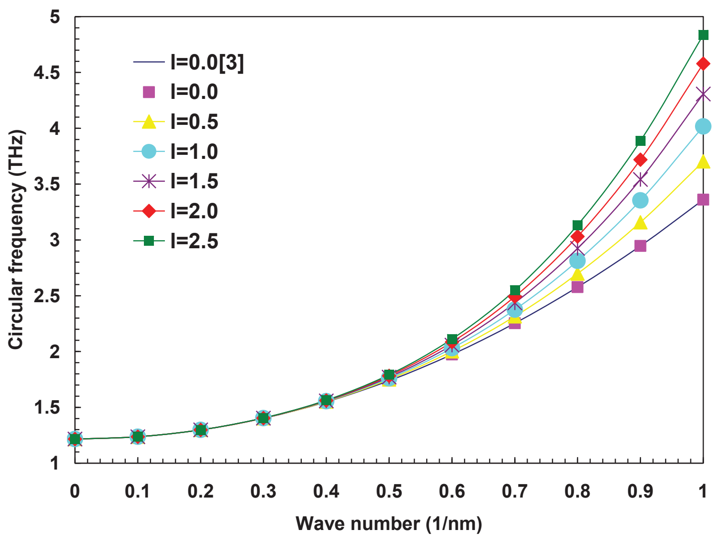

Fig. 1. (Color online) The effects of wave number and gradient parameter on the circular frequencies.

them by ignoring the wave number in x-direction.[2] In this table, the second type of cut-off frequencies is tabulated and the first type will be discussed in Table 2. From this table, it can be seen that with the increase of gradient parameter, the cut-off frequencies will increase. It is also found that increasing the wave number in y-direction will result in increase of the cut-off frequencies. From this table, it can be easily seen that the effects of wave number in y-direction are more than gradient parameter. In this table, our results are also verified with the results of Wang *et al.*[3] From our comparison, it is shown that the results of our methodology can be accurate for rectangular nanoplates. In Fig. 1, the influences of gradient

Table 2. The effects of halt wave number and elastic foundation on the cut-off frequencies.

	ω_c (II) K_w (10^{18})					ω_c (I) K_w (10^{18})				
K_y	0.0	1.0	2.0	3.0	4.0	0.0	1.00	2.0	3.0	4.0
0.1	0.224	1.165	1.632	1.993	2.298	0.0	1.143	1.617	1.980	2.287
0.2	0.455	1.231	1.680	2.032	2.331	0.0	1.143	1.617	1.980	2.287
0.3	0.700	1.341	1.762	2.100	2.391	0.0	1.143	1.617	1.980	2.287

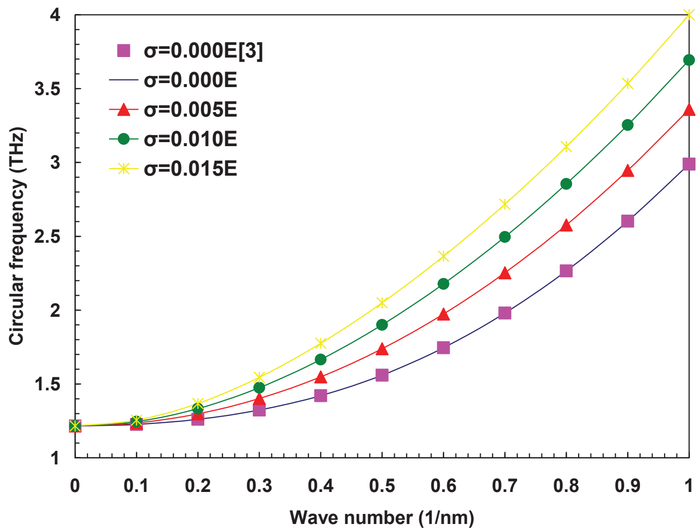

Fig. 2. (Color online) The effects of wave number and initial stress on the circular frequencies ($l = 0$).

parameter and wave number on the circular frequencies are presented. Our results are also compared with the circular frequencies available in the literature. From this figure, it is shown that with the increase of gradient parameter and wave number, the circular frequencies will increase. It is figured that it is also shown that the gradient parameter has significant effect on the wave propagation in rectangular nanoplates. It is mentioned that in this figure the effects of both initial stress and elastic matrix are also considered. In Figs. 2 and 3, the influences of initial stress and wave number on the circular frequencies are shown. In Fig. 2, the gradient parameter is neglected but in Fig. 3, the gradient parameter is assumed to be 1 nm^2. From these figures, one can easily find that the initial stress plays an important role in wave propagation in rectangular nanoplates. It is obtained that with the increase of initial stress, the circular frequencies will increase for both figures. It is also shown that the wave number has more effect for higher initial stresses. Table 2 illustrates the influences of elastic matrix on the cut-off frequencies. In this table the effects of Winkler foundation are investigated and the stiffness of shearing layer is assumed to be constant.

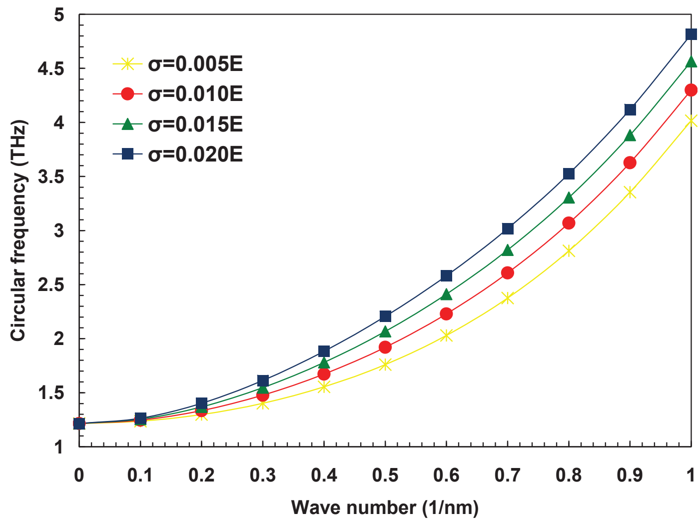

Fig. 3. (Color online) The effects of wave number and initial stress on the circular frequencies ($l = 1$ nm^2).

From this table, it is found that increasing the stiffness of Winkler foundation will result in increase of the cut-off frequencies. It is also shown that the second type of cut-off frequency is a bit more than first type. As it is expected, the first type of cut-off frequency is independent of half wave number in y-direction but second type increase with the increase of half wave numbers.

5. Conclusion

In this paper, in order to consider the small scale effects, strain gradient elasticity theory with one constant was used to investigate the wave propagation in rectangular nanoplates. The governing equation of motion with considering initial stresses and elastic matrix was derived on the basis of Kirchhoff plate theory. The Winkler foundation and the shearing layer were considered to model the elastic matrix. It was shown that with the increase of gradient parameter and wave number, the circular frequencies will increase. It was also obtained that with the increase of gradient parameter, the cut-off frequencies will increase. It could be seen that first type of cut-off frequency is independent of half wave number in y-direction but second type increase with the increase of half wave numbers.

References

1. M. Chen, J. Zang, D. Xiao and F. Liu, *J. Appl. Phys.* **105** (2009) 026102.
2. S. Narendar and S. Gopalakrishnan, *Compos. B* **43** (2012) 1275.
3. Y. Z. Wang, F. M. Li and K. Kishimoto, *Appl. Phys. A* **99** (2010) 907.
4. B. Arash, Q. Wang and K. M. Liew, *Comput. Methods Appl. Mech. Eng.* **1** (2012) 223.

5. H. Heirechea, A. Tounsib, A. Benzaira, M. Maachoua and E. A. A. Bedia, *Physica E* **40** (2008) 2791.
6. Y. Z. Wang, F. M. Li and K. Kishimoto, *Physica E* **42** (2010) 1356.
7. H. Liu and J. L. Yang, *Physica E* **44** (2012) 1236.
8. A. Besseghier, A. Tounsi, M. S. A. Houari, A. Benzair, L. Boumia and H. Heireche, *Physica E* **43** (2011) 1379.
9. S. Narendar and S. Gopalakrishnan, *Comput. Mater. Sci.* **47** (2009) 526.
10. S. Narendar and S. Gopalakrishnan, *Physica E* **42** (2010) 1601.
11. S. Narendar and S. Gopalakrishnan, *Physica E* **43** (2010) 423.
12. A. Assadi and B. Farshi, *Acta Mechanica* **222** (2011) 27.
13. T. P. Kaloni and U. Schwingenschlögl, *J. Appl. Phys.* **114** (2013) 184307.
14. T. P. Kaloni, M. U. Kahaly, R. Faccio and U. Schwingenschlögl, *Carbon* **64** (2013) 281.
15. N. Singh, T. P. Kaloni and U. Schwingenschlögl, *Appl. Phys. Lett.* **102** (2013) 023101.
16. Y. C. Cheng, T. P. Kaloni, Z. Y. Zhu and U. Schwingenschlogl, *Appl. Phys. Lett.* **101** (2013) 073110.
17. T. P. Kaloni and U. Schwingenschlög, *Chem. Phys. Lett.* **583** (2013) 137.
18. T. P. Kaloni, M. Tahir and U. Schwingenschlögl, *Sci. Rep.* **3** (2013) 3192.
19. J. Lei, Y. He, B. Zhang, Z. Gan and P. Zeng, *Int. J. Eng. Sci.* **72** (2013) 36.
20. S. Papargyri-Beskou and D. E. Beskos, *Arch. Appl. Mech.* **78** (2008) 625.
21. J. D. Achenbach, *Wave Propagation in Elastic Solids* (North-Holland, Oxford, 1973).