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Size-dependent bending analysis of Kirchhoff nano-plates based on a modified couple-stress theory including surface effects

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ABSTRACT

In the present work, a new Kirchhoff plate model is developed using a modified couple-stress theory to study the bending behavior of nano-sized plates, including surface energy and microstructure effects. The surface elasticity theory of Gurtin and Murdoch is used to model the surface energy effects, into the framework of the modified couple-stress theory of elasticity. Newtonian continuum mechanics approach is used to derive the differential form of the equilibrium equations for the modified Kirchhoff plate theory.

The modified plate rigidity is derived to express the size effects in nanoplates. Presence of a length scale parameter, in the context of the modified couple-stress theory, enables us to express the size effect in nano-scale structures. In addition, an intrinsic length scale parameter is determined as a result of taking surface energy effects into account.

In order to illustrate the model, an analytical solution of the static bending of a simply supported nano-plate has been derived. For ultra-thin plates it is noticed that the microstructure effects on bending rigidity and deflection, through the application of the modified-couple stress theory, is highly significant than that caused by the surface energy effect.

1. Introduction

Every physical theory possesses a certain domain of applicability outside which it fails to predict the physical phenomena with reasonable accuracy, Eringen [1]. For each theory, the domain of application defines the level of the considered constituents and the appropriate processes of interactions between these constituents. The components below this level would not be accounted for and consequently, the interaction process between these components and the other ones would be avoided also. As an obvious example, for a macro-scale body the surface component of the body is very small relative to the volume of the solid. Thus, we can neglect the surface as component of the continuum and focus our attention only on the bulk solid. For a tiny body the surface is very comparable to the bulk volume. Therefore, it should be taken into consideration and deserves to pay a considerable attention to its characteristics and the processes of interactions with the bulk of the continuum.

The same issue can be observed when we study the mechanical deformation of a macro-scale elastic continuum. In this case, it will be sufficient to investigate the behavior on the level of particles as already happened in the classical continuum mechanics theories, Truesdell and Noll [2]. On the contrary, for nano-scale systems we have to deal with the atomic discrete nature of the system. Thus, we have to concern primarily with the level of microstructure elements and investigate different interaction processes between those elements, Chen et al. [3]. Unfortunately, classical continuum mechanics is explicitly designed to be size-independent, which may call the applicability of classical continuum models on nanostructures into question. Several physical reasons may be ascribed to the breakdown of classical continuum mechanics at nano-scale size, Maraganti and Sharma [4].

The surface of a solid is considered as a region with a negligible thickness which has its own atom arrangement and properties differing from the bulk. Atoms at a free surface experience a different local environment than do atoms in the bulk of a solid material. As a result, the energy associated with these atoms will be different from that of the atoms in the bulk. The excess energy associated with surface atoms is surface free energy. For a solid with large size, such surface free energy is typically neglected because it is associated with only a few layers of atoms near the surface and the ratio of the volume occupied by the surface atoms and the total volume of material of interest is extremely small. However, for small solids with a comparable ratio of surface to

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bulk, the surface free energy effect is most likely significant. This is extremely true for nano-scale materials and structures.

Nevertheless, the presence of surface stress gives rise to a non-classical boundary condition which in combination with the constitutive relation of surface and the equations of classical elasticity forms a coupled system of field equations. This makes the solutions of the corresponding boundary value problem relatively difficult.

A generic mathematical model for the analysis of surface elasticity has been developed by Gurtin and Murdoch [5–8], where the surface stresses depend on deformations. The equilibrium and constitutive relations of the bulk solid are the same as those in classical elasticity, but the boundary conditions must ensure the force balance of the surface object. In Gurtin and Murdoch model, the surface is represented as a single layer combining of an infinite number of material particles as in classical elasticity, neglecting the microstructure of the surface. However, Guo and Zhao [9,10] considered the microstructure of the surface of nanofilms, where the surface consists of multi-layers of relaxed crystals. A lattice model is proposed where the possible bond relaxation of the atom is considered which alter the mechanical properties of the nano film.

Miller and Shenoy [11], developed a simple model based on the surface elasticity theory of Gurtin and Murdoch to determine the size effects on the elastic rigidities of nano-sized structural elements such as bars, beams and plates. Thus as the dimensions of the structure become smaller the presence of surfaces have to be accounted for in the modeling strategy. Most of surface effects, such as surface energy, surface tension and surface relaxation are studied by many investigators [9–16]. The effect of the residual stress-due to-surface tension on the bending behavior of nano-plates is studied by Wang and Zhao [12]. Moreover, the effect of surface relaxation in combining with surface tension on the bending behavior of nanobeams and plates is studied by Guo and Zhao [9,10].

The interactions at microscopic scale are the physical origin of many macroscopic phenomena. The fundamental departure of micro-continuum theories from the classical continuum theories is that the former is either a continuum model embedded with microstructures or a nonlocal model to describe the long-range material interaction, Chen et al. [3].

Any attempt to drop the continuity assumption in a modified theory is bounded to make the analysis extremely difficult and computationally intensive. Therefore there is a need for modified continuum theories that include new measures of deformation, which are length related, such as the curvature tensor. As a consequence, such a theory may also require the introduction of couple stresses, Hadjesfandiari and Dargush [17]. Cosserat and Cosserat [18] were the first to develop a mathematical model to analyze materials with couple stresses. In the original Cosserat theory, the kinematical quantities were the displacement and a material microrotation, which assumed to being independent of the continuum macrorotation.

Couple-stress theory is an extended continuum theory that includes the effects of a couple per unit area on a material volume, in addition to the classical direct and shear forces per unit area. This immediately admits the possibility of asymmetric stress tensor, since shear stress no longer have to be conjugate in order to ensure rotational equilibrium. Recently, Yang et al. [19] modified the classical couple stress theory and proposed a modified couple-stress model, in which the couple stress tensor is symmetrical and only one material length parameter is needed to capture the size effect which is caused by micro-structure.

Jomehzadeh et al. [20] developed a variational model for the vibration analysis of ultra-thin plates using the modified couple-stress theory and on the basis of Hamilton's principle. Tsiatas [21] studied the static bending analysis of isotropic micro-Kirchhoff plates using the modified couple-stress theory and on the basis of the principle of minimum potential energy. Ma et al. [22] developed a non-classical Mindlin plate model using the modified couple-stress theory and on the basis of Hamilton's principle.

On the other side, a general classical thin plate theory including surface effects, which can be used for static and dynamic analysis of plate-like thin film structures, was developed by Lu et al. [23]. The modeling of surface effects is based on the surface elasticity theory developed by Gurtin and Murdoch [5,7] and an additional material length scale parameter is determined. Moreover, Shaat et al. [13–15] developed a size-dependent model to study the static bending of Mindlin functionally graded plates incorporating surface energy effects based on Gurtin and Murdoch theory considering effects of surface tension.

The present study is focused on the presentation of a new Kirchhoff nanoplate model, based on the modified couple-stress theory of Yang et al. [19], and taking into account the surface energy and surface tension effects by using the surface elasticity theory of Gurtin and Murdoch. Classical Newtonian approach is used to derive the differential form of the equilibrium equations of the generalized Kirchhoff nanoplate.

The rest of the paper is organized as follows. In Section 2, the formulation of the equilibrium equations for the non-classical Kirchhoff plate model is developed using the modified couple stress theory (Yang et al. [19], Park and Gao [24]) and the surface elasticity theory of Gurtin and Murdoch [5,7]. Constitutive equations of the bulk and surface layer materials in addition to the kinematic equations of the Kirchhoff plate are presented in Section 3. Moreover, a length scale parameter, in the context of the modified couple stress theory, is presented to capture the size effect in nano-plates. Based on the equilibrium equations, constitutive relations and the kinematic equations; the equilibrium equations in terms of deflection are derived in the end of Section 3. To demonstrate the new proposed model, a simply supported plate problem is solved in Section 4, by applying the equilibrium equations derived in Section 3. Some numerical results are presented in Section 5 to show both the microstructure and surface energy effects on the bending rigidity of the plate. In addition, a parametric study is given to present the effect of surface parameters and the effect of the length scale parameter, mentioned in Section 3, on the bending behavior of simply supported Kirchhoff plates.

2. Equilibrium equations of the modified couple-stress plate theory including surface effects

The formulation of the equilibrium model for Kirchhoff plate including surface effects, in the framework of the modified couple stress theory, will be presented throughout this section. Surface energy and surface tension effects are handled through Gurtin and Murdoch model neglecting the microstructure of the surface. This formulation is developed on the basis of the classical Newtonian continuum mechanics, Reddy [25].

Consider an ultra-thin rectangle plate with uniform thickness h. A Cartesian coordinate system $x_i (i=1,\ 2,\ 3)$ is introduced so that the axes x_1 and x_2 are lying in the mid-plane of the plate, and the upper and lower surfaces S^+ and S^- of the plate are defined by $x_3 = \pm h/2$, respectively (see Fig. 1).

The differential form of the equilibrium equations for a size-dependent continuum, based on the modified couple-stress theory, is given by

$$\sigma_{ji,j} + f_i = 0 \tag{1}$$

$$\mu_{ji,j} + e_{ijk}\sigma_{jk} + C_i = 0 \tag{2}$$

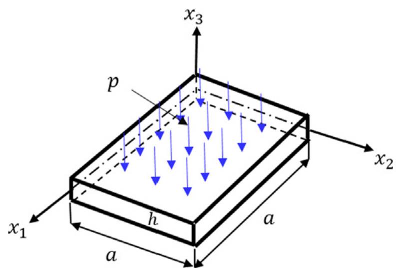

Fig. 1. Geometry of the plate.

where σ_{ji} , μ_{ji}, f_i , C_i and e_{ijk} denote force-stress, couple-stress, body force and body couple stress per unit volume, and the permutation tensor, respectively. In classical continuum mechanics, $\mu_{ji}=0$ and $C_i=0$. Therefore, angular equilibrium Eq. (2) shows that the force-stress tensor is symmetric.

The surface stresses on the two surfaces S^+ and S^- of the plate are denoted by $\tau^+_{i\alpha}$ and $\tau^-_{i\alpha}$, respectively, and satisfying the balance relations given by Gurtin and Murdoch [5,7]

$$\tau^+_{\beta i,\beta}-\sigma^+_{3i}=0 \text{ at } x_3=\frac{h}{2} \tag{3.a}$$

$$\tau^-_{\beta i,\beta}+\sigma^-_{3i}=0 \text{ at } x_3=-\frac{h}{2} \tag{3.b}$$

where $\sigma^+_{3i}=\sigma_{3i}(x_\beta\,,\,h/2)$ and $\sigma^-_{3i}=\sigma_{3i}(x_\beta,\,-h/2)$ are the bulk stresses at $x_3=\,\pm h/2$, respectively. In Eqs. (1)–(3.a) and (3.b) and throughout the paper, Latin subscripts range from the values 1 to 3, while Greek subscripts range over 1 and 2.

Since the thickness of the plate is very small relative to the other two dimensions, the governing equations can be integrated through the thickness to obtain the global plate equations. The resultant forces N_{ij}, the resultant moments M_{ij}, and the resultant couples Y_{ij} are defined as

$$N_{ij}=\int_{\frac{-h}{2}}^{\frac{h}{2}}\sigma_{ij}\,dx_3 \tag{4.a}$$

$$M_{ij}=\int_{\frac{-h}{2}}^{\frac{h}{2}}\sigma_{ij}\,x_3dx_3 \tag{4.b}$$

$$Y_{ij}\;=\;\int_{\frac{-h}{2}}^{\frac{h}{2}}\mu_{ij}dx_3 \tag{4.c}$$

Multiplying Eq. (1) by dx_3, and integrating through the thickness, we have

$$N_{\alpha i,\alpha}\,+\sigma^+_{3i}\,-\sigma^-_{3i}\,+P_i=0 \tag{5}$$

where $P_i=\int_{-h/2}^{h/2}f_i dx_3$.

Furthermore, multiplying Eq. (1) by x_3dx_3 and integrating through the thickness for $i\,=1$ and 2 and notice that the integration for the case of index $i=3$ has no physical application, thus, it is omitted in the rest of derivation.

$$M_{\alpha\beta,\alpha}\,+\frac{h}{2}\,(\sigma^+_{3\beta}+\sigma^-_{3\beta}\,)-N_{\beta 3}+r_\beta\;=0 \tag{6}$$

where $r_\beta=\int_{-h/2}^{h/2}f_\beta x_3dx_3$.

Here in this paper, only, the contribution of the transverse applied load on the plate deflection is considered. Consequently, $f_\beta=0$ and, only, $P_3=\int_{-h/2}^{h/2}f_3\;dx_3\neq 0$ and $r_\beta=0$.

Substituting the surface balance relations (3.a) and (3.b) into (4.a)–(4.c)–(6), the governing equations of the plate including the surface effects are defined as

$$N_{\alpha i,\alpha}+\tau^+_{\alpha i,\alpha}+\,\tau^-_{\alpha i,\alpha}+P_i=0 \tag{7.a}$$

$$M_{\alpha\beta,\alpha}+\frac{h}{2}(\tau^+_{\alpha\beta,\alpha}-\,\tau^-_{\alpha\beta,\alpha})-N_{\beta 3}=0 \tag{7.b}$$

Consequently, the generalized resultant forces and resultant moments for plate incorporating surface energy effects are

$$N^*_{\alpha i}=N_{\alpha i}+\tau^+_{\alpha i}+\,\tau^-_{\alpha i} \tag{8.a}$$

$$M^*_{\alpha\beta}=M_{\alpha\beta}+\frac{h}{2}(\tau^+_{\alpha\beta}-\,\tau^-_{\alpha\beta}) \tag{8.b}$$

The equilibrium Eq. (7) can be further written as

$$N^*_{\alpha i,\alpha}+P_i=0 \tag{9.a}$$

$$M^*_{\alpha\beta,\alpha}-N_{\beta 3}=0 \tag{9.b}$$

Using Eq. (7.a) and by simple manipulations, Eq. (9.b) can be reformulated as

$$M^*_{\alpha\beta,\alpha\beta}+\tau^+_{\beta 3,\beta}+\tau^-_{\beta 3,\beta}+P_3=0 \tag{9.c}$$

Equations (9.a)–(9.c) are the general equilibrium equations of the classical plate theory including surface effects.

Assuming zero in-plane displacements of the mid-plane of the plate and also, assume the body couple stress per unit volume $C_i=0$ in addition to $\mu_{i3}=\mu_{3i}=0$, therefore, the couple-stress equilibrium equation, Eq. (2), can be expressed as

$$\mu_{\alpha i,\alpha}\,+e_{ijk}\sigma_{jk}=0 \tag{10}$$

Integrate Eq. (10) through the thickness we obtain

$$Y_{11,1}\,+Y_{21,2}+N_{23}\,-N_{32}\;=0 \tag{11.a}$$

$$Y_{12,1}\,+Y_{22,2}\,-N_{13}+N_{31}\;=0 \tag{11.b}$$

Differentiate Eq. (11.a) w.r.t. x_2, and Eq. (11.b) w.r.t. x_1, and subtract we obtain the following

$$\frac{\partial^2 Y_{11}}{\partial x_1\partial x_2}+\frac{\partial^2 Y_{21}}{\partial x_2^2}-\frac{\partial^2 Y_{12}}{\partial x_1^2}-\frac{\partial^2 Y_{22}}{\partial x_1\partial x_2}+\frac{\partial N_{23}}{\partial x_2}-\frac{\partial N_{32}}{\partial x_2}+\frac{\partial N_{13}}{\partial x_1}-\frac{\partial N_{31}}{\partial x_1}=0. \tag{12}$$

Differentiate Eq. (9.b) w.r.t. x_β and add to Eq. (7.a) with index $i=3$ we obtain

$$\frac{\partial^2 M^*_{11}}{\partial x_1^2}+2\frac{\partial^2 M^*_{12}}{\partial x_1\partial x_2}+\frac{\partial^2 M^*_{22}}{\partial x_2^2}-\frac{\partial N_{31}}{\partial x_1}-\frac{\partial N_{32}}{\partial x_2}+\frac{\partial N_{13}}{\partial x_1}+\frac{\partial N_{23}}{\partial x_2}+\frac{\partial \tau^+_{13}}{\partial x_1}+\frac{\partial \tau^+_{23}}{\partial x_2}+\frac{\partial \tau^-_{13}}{\partial x_1}+\frac{\partial \tau^-_{23}}{\partial x_2}+P_3=0 \tag{13}$$

Subtract Eq. (12) from Eq. (13)

$$\frac{\partial^2 M^*_{11}}{\partial x_1^2}+2\frac{\partial^2 M^*_{12}}{\partial x_1\partial x_2}+\frac{\partial^2 M^*_{22}}{\partial x_2^2}-\frac{\partial^2 Y_{11}}{\partial x_1\partial x_2}-\frac{\partial^2 Y_{21}}{\partial x_2^2}+\frac{\partial^2 Y_{12}}{\partial x_1^2}+\frac{\partial^2 Y_{22}}{\partial x_1\partial x_2}+\frac{\partial \tau^+_{13}}{\partial x_1}+\frac{\partial \tau^+_{23}}{\partial x_2}+\frac{\partial \tau^-_{13}}{\partial x_1}+\frac{\partial \tau^-_{23}}{\partial x_2}+P_3=0 \tag{14}$$

Eq. (14) represents the equilibrium equation of the modified Kirchhoff plate, including surface effects, based on the modified couple-stress theory in terms of the resultant moments, resultant couples and surface stresses. Further, we have to notice that M^*_{ij} represent not only the resultant moments of the bulk material stresses but taking into account the contribution of surface stresses as given by Eq. (8.b). Therefore, it is obviously clear that the surface stresses contribute to the equilibrium of the plate by an additional normal force and part of the resultant moment components.

3. Equilibrium equations in terms of deflection

The force-stress σ_{ij} and the deviatoric part of the couple-stress μ_{ij} are defined in terms of the strain ε_{ij} and the symmetric curvature tensor χ_{ij}, respectively, as

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} \tag{15.a}$$

$$\mu_{ij} = 2\mu l^2 \chi_{ij} \tag{15.b}$$

where λ and μ are Lame's coefficients and ℓ is the material length scale parameter which is regarded as a material property measuring the effect of couple stress (Mindlin [26]). This parameter can be determined from torsion tests of slim cylinders (Chong et al. [27]) or bending tests of thin beams (Lam et al. [28]).

The strain tensor ε_{ij} and the curvature tensor χ_{ij} are expressed in terms of displacement vector u_i and rotation vector θ_i as

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j}+u_{j,i}) \tag{16.a}$$

$$\chi_{ij} = \frac{1}{2}(\theta_{i,j}+\theta_{j,i}) \tag{16.b}$$

where the rotation vector can be expressed in terms of displacement vector as

$$\theta_i = \frac{1}{2} e_{ijk}\, u_{k,j} \tag{16.c}$$

The constitutive relations of the surface layers S^+ and S^- are expressed by Gurtin and Murdoch [5,7] as

$$\tau^{\pm}_{\alpha\beta} = \tau^{\pm}_0\ \delta_{\alpha\beta} + (\mu^{\pm}_0 - \tau^{\pm}_0)(u^{\pm}_{\alpha,\beta}+u^{\pm}_{\beta,\alpha}) + (\lambda^{\pm}_0 + \tau^{\pm}_0)u^{\pm}_{\gamma,\gamma}\delta_{\alpha\beta} + \tau^{\pm}_0\, u^{\pm}_{\alpha,\beta} \tag{17.a}$$

$$\tau^{\pm}_{\alpha 3} = \tau^{\pm}_0\ u^{\pm}_{3,\alpha} \tag{17.b}$$

where $\tau^{\pm}_0$ are the residual surface tensions under unconstrained conditions, $\lambda^{\pm}_0$ and $\mu^{\pm}_0$ are the surface Lame's constants on the surface S^+ and S^-, respectively. If the top and bottom layers have same material properties, the stress strain relations reduce to

$$\tau^{\pm}_{\alpha\beta} = \tau_0\ \delta_{\alpha\beta} + (\mu_0 - \tau_0)(u^{\pm}_{\alpha,\beta}+u^{\pm}_{\beta,\alpha}) + (\lambda_0 + \tau_0)u^{\pm}_{\gamma,\gamma}\delta_{\alpha\beta} + \tau_0 u^{\pm}_{\alpha,\beta} \tag{18.a}$$

$$\tau^{\pm}_{\alpha 3} = \tau_0\ u^{\pm}_{3,\alpha} \tag{18.b}$$

In Eqs. (18.a) and (18.b), the terms ($\tau_0 u^{\pm}_{\alpha,\beta}$ and $\tau_0 u^{\pm}_{3,\alpha}$) are introduced as a consequence of exploiting the Lagrangian surface description and considering the pre-strain developed at the plate surface (Ru [29], Shaat et al. [16]). In most previous works (Wang et al. [30], Zang and Zhao [31], Mogilevskaya et al. [32]), theoretical analyses were based on the Eulerian surface elasticity, in which the out-plane terms of surface stress were neglected and the effect of residual stress in the bulk was not taken into account. As an illustration, in this paper, we will consider the effects of these factors on the size-dependent behavior of nano-plates, which are not considered by some previous authors.

According to the basic hypothesis of Kirchhoff plate theory and ignoring the in-plane displacements of the mid-plane of the plate, the displacements field of the plate may be expressed as

$$u_1(x_1,x_2,x_3) = -x_3\frac{\partial w(x_1,x_2)}{dx_1},\ u_2(x_1,x_2,x_3) = -x_3\frac{\partial w(x_1,x_2)}{dx_2},\ u_3(x_1,x_2,0) = w(x_1,x_2) \tag{19}$$

where w is the deflection of the mid-plane of the plate.

Under the assumption of small deformation and linear strain-displacement relations, from Eq. (16.a) the strain components of the Kirchhoff plate can be expressed as

$$\varepsilon_{\alpha\beta} = -x_3\, w_{,\alpha\beta} \tag{20}$$

From Eqs. (16.b) and (16.c), the components of the rotation vector θ_i and the curvature tensor χ_{ij} can also be expressed in terms of the displacement field as

$$\theta_1 = \frac{\partial w}{\partial x_2},\ \theta_2 = -\frac{\partial w}{\partial x_1},\ \theta_3 = 0 \tag{21}$$

$$\chi_{11} = \frac{\partial^2 w}{\partial x_1 \partial x_2},\ \chi_{12} = \chi_{21} = \frac{1}{2}\left(\frac{\partial^2 w}{\partial x_2^2} - \frac{\partial^2 w}{\partial x_1^2}\right),\ \chi_{22} = -\frac{\partial^2 w}{\partial x_1 \partial x_2} \tag{22}$$

$$\chi_{13} = \chi_{31} = \chi_{23} = \chi_{32} = \chi_{33} = 0 \tag{23}$$

We can easily noticed, from Eq. (15.b), that

$$\mu_{13} = \mu_{23} = \mu_{31} = \mu_{32} = \mu_{33} = 0 \tag{24}$$

By using Eqs. (4.b), (4.c), (15.a), (15.b), (16.a)–(16.c), (18.a), and (18.b), the resultant moments $M^*_{\alpha\beta}$ and resultant couples $Y_{\alpha\beta}$ are given as

$$M^*_{11} = -D^{b-s}\left(\frac{\partial^2 w}{\partial x_1^2} + \nu\frac{\partial^2 w}{\partial x_2^2}\right) \tag{25.a}$$

$$M^*_{22} = -D^{b-s}\left(\nu\frac{\partial^2 w}{\partial x_1^2} + \frac{\partial^2 w}{\partial x_2^2}\right) \tag{25.b}$$

$$M^*_{12} = -\frac{(1-\nu)}{2}D^{b-s}\frac{\partial^2 w}{\partial x_1 \partial x_2} \tag{25.c}$$

$$Y_{11} = 2\mu h l^2 \frac{\partial^2 w}{\partial x_1 \partial x_2} \tag{25.d}$$

$$Y_{22} = -2\mu h l^2 \frac{\partial^2 w}{\partial x_1 \partial x_2} \tag{25.e}$$

$$Y_{12} = \mu h l^2\left(-\frac{\partial^2 w}{\partial x_1^2} + \frac{\partial^2 w}{\partial x_2^2}\right) \tag{25.f}$$

where $D^{b-s} = D\,(1+\eta/h)$ is the bending rigidity of the classical Kirchhoff nanoplate including only surface effects, $D = Eh^3/12(1-\nu^2)$ is the bending rigidity of the classical Kirchhoff plate. The scale factor parameter η is the ratio between the surface properties and the bulk properties to determine the significance of surface energy effects and is explicitly given by (Lu et al. [23])

$$\eta = \frac{1}{E}[6(1-\nu^2)(\mu_0+\lambda_0+\tau_0) - 2\nu(1+\nu)\tau_0] \tag{25.g}$$

Substitute the terms given by Eqs. (25.a)–(25.g) into equilibrium Eq. (14) we obtain the equilibrium equation of the plate in terms of the deflection as

$$2\tau_0\left(\frac{\partial^2 w}{\partial x_1^2} + \frac{\partial^2 w}{\partial x_2^2}\right) - D^{b-s-c}\,\nabla^4 w + P_3 = 0 \tag{26}$$

where the bending rigidity of the Kirchhoff nanoplate, including surface energy and microstructure effects, is expressed as

$$D^{b-s-c} = (D^{b-s} + \mu h l^2) \tag{27.a}$$

Neglecting surface energy effects, the bending rigidity including only microstructure effects will be

$$D^{b-c} = (D + \mu h l^2) \tag{27.b}$$

It is obviously noticed that the surface energy and the microstructure effects have a significant contribution on the bending rigidity of the plate. Furthermore, the surface effects add a new normal force on the equilibrium system of the plate as a consequence of considering the effects of the residual stress (τ_0), as shown in Eq. (26).

4. Analytical solution for simply supported rectangular plate

To illustrate the modified Kirchhoff plate model, given in the preceding sections, the problem of a simply supported nanoplate is solved. We assume the nanoplate is a square of side length a and nano-thickness h (see Fig. 1).

Let us assume that the plate is subjected to double Fourier sinusoidal loading

$$P_3(x,y) = q_0 \sin\frac{\pi x}{a} \sin\frac{\pi y}{a} \tag{28}$$

where q_0 is the intensity of the mechanical load. Since the plate is simply supported, the boundary conditions along edges can be written as

$$w=0, \quad -M^*_{11} - Y_{12} = 0 \text{ along the edges } x_1 = 0 \text{ and } x_1 = a, \tag{29.a}$$

$$w=0, \quad -M^*_{22} + Y_{12} = 0 \text{ along the edges } x_2 = 0 \text{ and } x_2 = a \tag{29.b}$$

Substitute Eqs. (25.a, 25.b and 25.f) into the boundary conditions given by Eqs. (29.a) and (29.b), they can be expressed in terms of deflection as

$$w=0, \quad A\frac{\partial^2 w}{\partial x_1^2} + B\frac{\partial^2 w}{\partial x_2^2} = 0, \text{ along the edges } x_1 = 0 \text{ and } x_1 = a, \tag{30.a}$$

$$w=0, \ B\frac{\partial^2 w}{\partial x_1^2} + A\frac{\partial^2 w}{\partial x_2^2} = 0, \text{ along the edges } x_2 = 0 \text{ and } x_2 = a. \tag{30.b}$$

where $A = (D^{b-s} + \mu h l^2)$ and $B = (\nu D^{b-s} - \mu h l^2)$.

Suppose the deflection is distributed over the mid-plane according to a function satisfying the boundary conditions given by Eqs. (30.a) and (30.b) such as the following function

$$w(x,y) = C \sin\frac{\pi x}{a} \sin\frac{\pi y}{a} \tag{31}$$

It is obviously clear that this proposed solution satisfying all boundary conditions given by Eqs. (30.a) and (30.b). Thus, the amplitude C of the deflection function can be easily obtained by substituting the deflection function into the equilibrium Eq. (26) as

$$C = \frac{q_0 a^2}{4\pi^2[\tau_0 + D^{b-s-c}(\pi/a)^2]} \tag{32}$$

5. Numerical results

Here in this section, some numerical examples are presented for simply supported square nano-plate to illustrate the surface energy and the microstructure effects on the plate rigidity and deflection. Consider the square plate solved in Section 4. The plate is expected to be made of aluminum and the material parameters of the plate are, as given by Gao and Mahmoud [33]

$$E = 90\times10^9 N/m^2,\ \nu = 0.23,\ \lambda_0 = 3.4939 N/m,$$

$$\mu_0 = -5.4251 N/m,\ \tau_0 = 0.5689 N/m,\ l = 6.58\times10^{-6}\text{m}.$$

For the given material constants and thickness $h = 0.2$ nm, the size-dependent scale parameter η has the value (-8.96×10^{-11}). Consequently, the classical plate rigidity including only surface effects D^{b-s}) will be changed to (0.552 D), which obviously means that the rigidity of the plate is reduced as a result of the surface energy effects. We have to mention that surface energy effects may increase or reduce the plate rigidity depending on the elastic constants of the surface material.

As the result of applying the modified couple-stress theory and taking into account the surface energy effects, the overall bending rigidity of the plate D^{b-s-c} will be changed to $10^{10}D$. Obviously, for ultra-thin plates the microstructure effect is highly dominant than that caused by the surface energy effect. Using Eq. (32), the amplitude C of the simply supported plate deflection, subjected to harmonic load given by Eq. (28), is $C/(q_0 a^4/4\pi^4 D) = 9.9985\times10^{-11}$.

To represent the surface energy and the microstructure effects on the plate rigidity, the non-dimensional difference between the plate bending stiffnesses predicted by the modified Kirchhoff model and the classical Kirchhoff model ($(D^{b-s-c} - D)/D$) is shown in Fig. 2. The figure shows that the contribution of the surface energy is to reduce the plate stiffness, while the contribution of the microstructure deformation is to provide an additional significant stiffness for ultra-thin plates. Moreover, the figure shows the effect of the residual stress τ_0 on the plate rigidity.

Fig. 3 shows the non-dimensional difference between the plate deflections predicted by the modified Kirchhoff model and the classical Kirchhoff model. The effect of microstructure couple stress is set on at micro scale thicknesses, while surface energy effects are launched at nano-scale thicknesses. The plate provides a negative non-dimensional difference in deflection when considering the effect of couple stress for microscale thicknesses

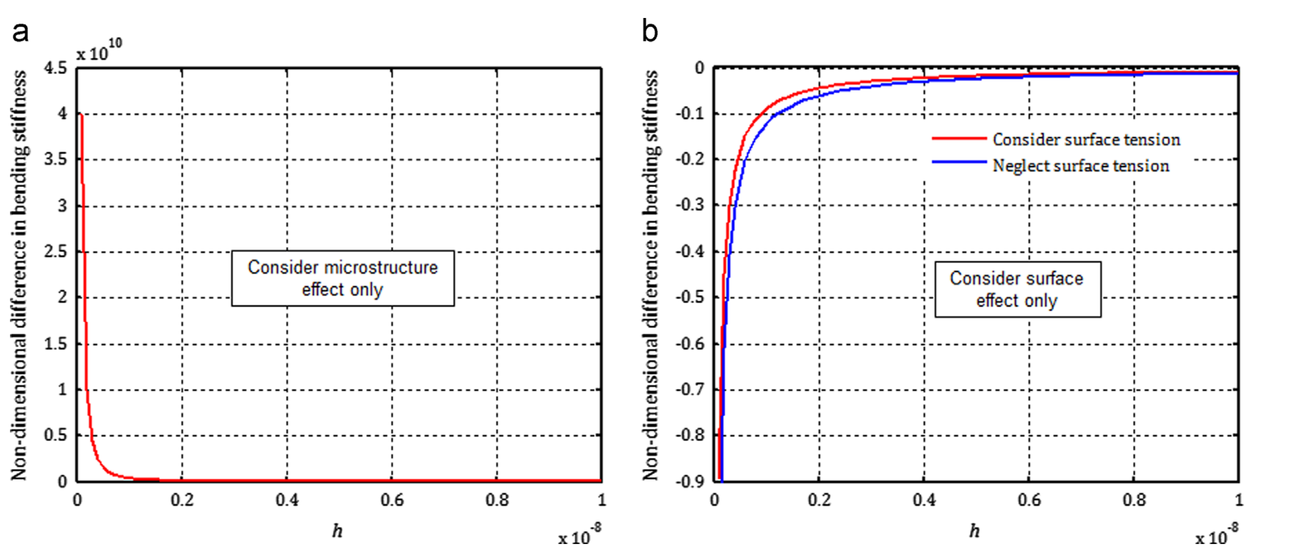

Fig. 2. Non-dimensional difference between the plate bending stiffnesses predicted by the modified Kirchhoff and the classical Kirchhoff (a) couple stress effect ($(D^{b-c} - D)/D$) and (b) surface energy effect ($(D^{b-s} - D)/D$).

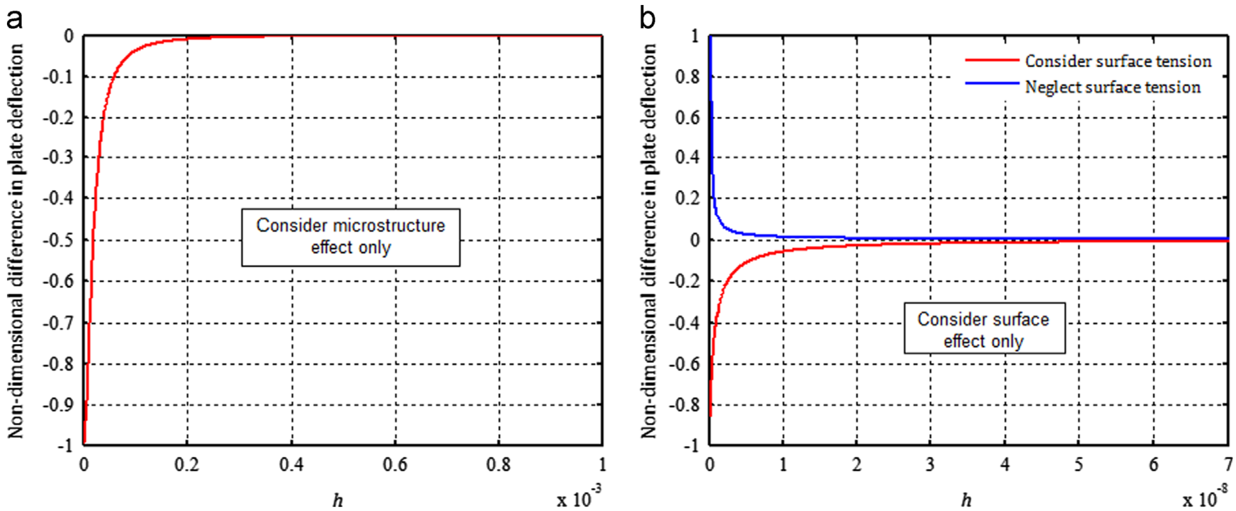

Fig. 3. Non-dimensional difference between the plate deflections predicted by the modified Kirchhoff and the classical Kirchhoff (a) couple stress effect ($(w^{b-c}-w)/w$) and (b) surface energy effect ($(w^{b-s}-w)/w$)).

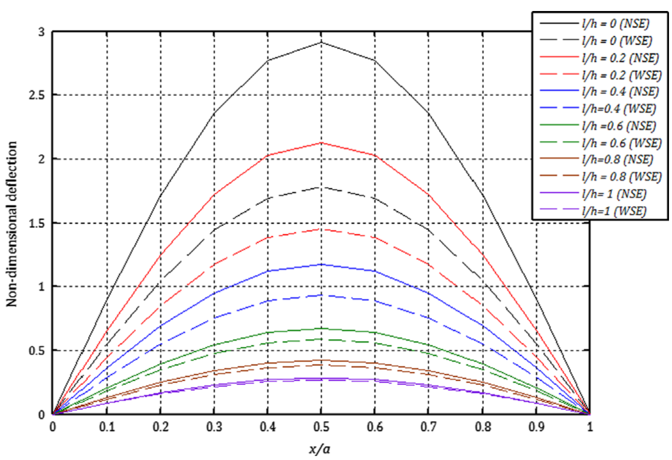

Fig. 4. Non-dimensional deflection for different l/h ratios (*NSE: neglect surface effects; *WSE: consider surface effects).

where surface energy effects cannot be observed. Moreover, it is noticed that the plate provides a reversed behavior when considering or neglecting the residual stress effects. Based on Fig. 2 the plate must provide a positive non-dimensional deflection. However, the added normal force in Eq. (26), due to the residual stress τ_0, has a great contribution on the plate deflection higher than that of the plate rigidity itself.

From Figs. 2 and 3, surface energy and microstructure effects may increase or reduce the plate rigidity depending on the elastic constants of the surface and the material scale parameter (l).

The effect of surface energy and microstructure on the plate behavior is well shown in Fig. 4. The figure shows the non-dimensional deflection distribution $[\varpi=(100Eh^3/q_0a^4)\times\omega(x,a/2)]$ along the plate length for different l/h ratios for plate thickness of $h=1\times10^{-9}$m. The figure shows that by increasing the ratio l/h, the couple stress contribution increases and consequently the plate deflection decreases. Moreover, considering the effect of surface residual stress τ_0 stiffens the plate and reduces its deflection, as a consequence.

6. Conclusion

In the present work a new model for bending of Kirchhoff nanoplates incorporating surface energy effects is developed in the framework of the modified couple stress theory. Unlike the classical plate theories, the proposed model captures the size effects of ultra-thin plates by introducing a new length scale parameter to account for the microstructural effect of the bulk material. In addition, an intrinsic length scale parameter is introduced as a result of taking surface energy effect into account. Direct Newtonian approach has been used to derive the equilibrium equations of the modified Kirchhoff plate. The modified couple-stress theory is used to express for the microstructure effect and surface energy effect is taking into account by using the surface elasticity theory developed by Gurtin and Murdoch.

One example of bending analysis for a simply supported ultra-thin plate is solved to illustrate the model. The effect of size length scale parameters (η and l) is significant as the thickness becomes very small. The length scale introduced by the modified couple-stress theory leads to a dominant increasing of the bending rigidity of the plate. The intrinsic length scale parameter resulting from taking surface energy into account has a less significant effect comparing to that given by the microstructure. Meanwhile, surface energy effects may increase or reduce bending rigidity of the plate depending on the surface material constants. Moreover, for ultra-thin plates the surface tension has a greater effect than the plate rigidity on the plate behavior.

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