European Journal of Mechanics A/Solids 40 (2013) 50–59

Contents lists available at SciVerse ScienceDirect

European Journal of Mechanics A/Solids

journal homepage: www.elsevier.com/locate/ejmsol

A micro-scale modeling of Kirchhoff plate based on modified strain-gradient elasticity theory

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ARTICLE INFO

Article history:
Received 9 October 2012
Accepted 23 December 2012
Available online 11 January 2013

Keywords:
Modified strain-gradient theory
Kirchhoff plate
Extended Kantorovich method
Size effect

ABSTRACT

A Kirchhoff micro-plate model is presented based on the modified strain gradient elasticity theory to capture size effects, in contrast with the classical plate theory. The analysis is general and can be reduced to the modified couple stress plate model or classical plate model once two or all material length scale parameters in the theory are set zero respectively. Governing equation and boundary conditions of an isotropic rectangular micro-plate are derived using minimum potential energy principle. Various boundary conditions including simply supported and clamped edges are covered by the analysis. The extended Kantorovich method (EKM) which is an accurate approximate closed-form solution is applied to solve the resulting sixth order boundary value problem. Application of EKM to the partial differential equation (PDE) yields two ordinary differential equations (ODEs) in the independent *x* and *y* coordinates. The resulted ODEs are solved in an iterative manner. Exact closed-form solutions are presented for both ODEs in all of the iteration. It is shown that the method provides accurate predictions with very fast convergence. Numerical results reveal that the differences between the deflection predicted by the modified strain gradient model, the couple stress model and the classical model are large when the plate thickness is small and comparable to the material length scale parameters. However, the differences decrease with increasing the plate thickness. Validation of the presented EKM solution shows good agreement with available literature.

1. Introduction

It has been experimentally demonstrated that the micro scale structures are size-dependent. For example, it has been observed in some metals which are deformed plastically (Guo et al., 2005; Poole et al., 1996). In the micro-torsion test, Fleck et al. (1992) observed that the torsional hardening of thin copper wires increases when the wires diameter decreases. Also researchers have proven size-dependent behavior in some polymers. For instance, Chong and Lam (1999) observed strong size-dependency in epoxy and Lam et al. (2003) investigated size-dependency in epoxy polymeric beams and their results show a significant enhancement of normalized bending rigidity as the thickness of the beam decreases. In the micro-bending test of polypropylene micro-cantilevers, McFarland and Colton (2005) showed a significant difference between their results and values predicted by the classical theory of beam. The aforementioned experimental works reveal that the intrinsic behavior of some materials is size-dependent and the classical theory cannot predict reliable results due to lack of material length scale parameters while the size of structures is at micron-scale. Consequently, some higher-order theories have been proposed to take into account the size effect in which constitutive equations involve length scale parameters as well as classical Lame's constants.

One of the higher-order continuum theories is classical couple stress theory proposed by some investigators such as Toupin (1962), Mindlin and Tiersten (1962) and Koiter (1964). The theory introduces two material length scale parameters for an isotropic elastic material. The classical couple stress theory has been employed in some static and dynamic problems (Zhou and Li, 2001; Kang and Xi, 2007). Yang et al. (2002) suggested a modified couple stress theory in which a higher-order equilibrium equation, i.e. the equilibrium equation of couple of couples, is considered. As a result, the couple stress tensor should be symmetric and only symmetric part of rotation gradient tensor contributes to storage of elastic energy. Therefore, one material length scale parameter associated with the symmetric rotation gradient tensor is only included in constitutive equations in addition to two classical constants. The theory has been applied to study static and dynamic behavior of size-dependent Bernoulli–Euler and Timoshenko beam models by

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0997-7538/$ – see front matter
http://dx.doi.org/10.1016/j.euromechsol.2012.12.008

some researchers such as Park and Gao (2006), Kong et al. (2008), Ma et al. (2008), Asghari et al. (2010, 2011) and Reddy (2011). In the analysis of plates employing the modified couple stress theory, Tsiatas (2009) derived the governing equation of Kirchhoff plate with the most general form of boundary conditions and Jomehzadeh et al. (2011) studied the size-dependent vibration analysis of Kirchhoff plate.

Another higher-order continuum theory has been developed by Mindlin (1965) in which strain energy is considered as a function of first and second-order gradients of strain tensor. In a normal case, the theory involving only first-order gradient of strain tensor introduces five new constants as well as Lame's constants for an isotropic linear elastic material (Mindlin and Eshel, 1968). Altan and Aifantis (1992) proposed a simplified strain gradient theory involving only one new constant. Lazopoulos (2004) formulated a geometrically nonlinear size-dependent plate based on the simplified strain gradient elasticity theory. Fleck and Hutchinson (1993, 1997 and 2001) reformulated the Mindlin's theory and called it the strain gradient theory. Lam et al. (2003) utilizing the higher-order equilibrium equation suggested by Yang et al. (2002) modified the strain gradient elasticity theory. The theory involves three material length scale parameters corresponding to the dilatation gradient tensor, the deviatoric stretch gradient tensor and the symmetric rotation gradient tensor. The higher-order stresses are defined as the work-conjugate to the higher-order deformation metrics. It should be noted that the modified strain gradient elasticity theory can be reduced to the modified couple stress theory if two of the three material length scale parameters are taken to be zero. In other words, the modified couple stress theory is a special case of the modified strain gradient elasticity theory. The modified strain gradient elasticity has been utilized to investigate the static and dynamic response of size-dependent Bernoulli–Euler and Timoshenko beam models by some researchers such as Kong et al. (2009) and Wang et al. (2010). Buckling of axially loaded micro-scaled beams based on both of the modified couple stress theory and the modified strain gradient elasticity theory has been studied by Akgoz and Civalek (2011). Based on the simplified form of the Mindlin's strain gradient theory, a variational analysis of both rectangular and circular plated has been carried out by Papargyri-Beskou et al. (2010). Moreover, a new formulation based on the modified strain gradient elasticity theory has been developed by Wang et al. (2011) for simply supported plates. However, two misconceptions have occurred in the study concerning stress–strain relation and also extracting boundary conditions. It should be noted that the proper boundary conditions, which are derived in the presented work, are not satisfied by the double Fourier' series assumed in the Eq. (33) of the paper (Wang et al., 2011) for the static and dynamic analysis. Therefore the obtained results in both of the static and dynamic analysis would not be correct, naturally.

On the other hand, in the categories of numerical procedures, the Extended Kantorovich Method (EKM) has been first introduced by Kerr (1969) using the idea of the Kantorovich method to obtain highly accurate closed-form solution for torsion of prismatic bars with rectangular cross-section. Since then, EKM has been extensively used in many applications. For instance, one is referred to eigenvalue problems (Kerr, 1969), buckling (Yuan and Jin, 1998) and free vibrations (Dalaei and Kerr, 1996) of thin rectangular plates, bending of thick rectangular isotropic (Aghdam et al., 1996; Yuan et al., 1998) and orthotropic (Aghdam and Falahatgar, 2003) plates, free-edge strength analysis (Kim et al., 2000), vibration of variable thickness plates (Shufrin and Eisenberger, 2006) and buckling of symmetrically laminated plates (Ungbhakorn and Singhatanadgid, 2006). Although the extended Kantorovich method is based on the variational principle, it has been shown that initial guess functions are not required to satisfy the boundary conditions (Kerr and Alexander, 1961; Dalaei and Kerr, 1995; Aghdam et al., 1996). Utilizing the proposed method reduces the problem of solving a partial differential equation to a set of ordinary differential equations in the x and y directions. Iterative scheme of the method forces the solution to satisfy all boundary conditions. These two features make the EKM more appropriate than the traditional weighted residual methods such as Galerkin or Ritz method. Furthermore, the strain gradient plate models are described by a sixth order differential equation. Thus, the FEM conformity requirements demand elements of C^2 continuity which makes FEM method tedious and impractical for the problem.

The object of the present work is to provide a solution for bending analysis of a rectangular micro scale Kirchhoff plate using the modified strain gradient elasticity theory and variational principle. For this purpose, a highly accurate method, i.e. the EKM is adopted to solve the energy based derived six order PDE together with the appropriate boundary conditions. The outline of this paper is organized as follows. In Section 2, the variational formulation of the micro scale Kirchhoff plate based on the strain gradient elasticity theory is in detail deduced using the minimum potential energy principle. Then governing equation and boundary conditions are obtained simultaneously. In Section 3, the extended Kantorovich method is implemented. Subsequently, in Section4 the static bending problem for both simply supported and clamped boundary conditions is solved and numerical results of the current Kirchhoff plate model are compared with both of the classical and modified couple stress model. Validation of the presented EKM is also carried out via the available literature. Finally, some conclusions are summarized in Section 5.

2. Governing equation of micro plate

The strain gradient elasticity theory introduces dilatation gradient tensor and the deviatoric stretch gradient tensor as well as the symmetric rotation gradient. The strain energy U for an isotropic linear elastic material occupying region V based on the modified strain gradient elasticity theory is written as (Lam et al., 2003)

$$U = \frac{1}{2}\int_V \left(\sigma_{ij}\varepsilon_{ij} + p_i\gamma_i + \tau^{(1)}_{ijk}\eta^{(1)}_{ijk} + m_{ij}\chi^S_{ij}\right)\mathrm{d}v \tag{1}$$

where

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{2}$$

$$\gamma_i = \varepsilon_{mm,i} \tag{3}$$

$$\eta^{(1)}_{ijk} = \eta^S_{ijk} - \frac{1}{5}\left(\delta_{ij}\eta^S_{mmk} + \delta_{jk}\eta^S_{mmi} + \delta_{ki}\eta^S_{mmj}\right) \tag{4}$$

$$\chi^S_{ij} = \frac{1}{4}\left(e_{imn}u_{n,mj} + e_{jmn}u_{n,mi}\right) \tag{5}$$

in which comma indicates partial derivative and u_i is the displacement vector, ε_{ij} is the strain tensor, γ_i is the dilatation gradient vector, $\eta^{(1)}_{ijk}$ is the deviatoric stretch gradient tensor, χ^S_{ij} is the symmetric rotation gradient tensor, δ_{ij} is the Kronocker delta, e_{ijk} is the permutation symbol and η^S_{ijk} is the symmetric part of second-order displacement gradient tensor defined by

$$\eta^S_{ijk} = \frac{1}{3}\left(u_{i,jk} + u_{j,ki} + u_{k,ij}\right) \tag{6}$$

Both of the tensors $\eta_{ijk}^{(1)}$ and χ_{ij}^{S} are symmetric with respect to all of the subscripts elucidated by

$$\eta_{ijk}^{(1)} = \eta_{jik}^{(1)} = \eta_{ikj}^{(1)} = \eta_{kji}^{(1)}, \quad \chi_{ij}^{S} = \chi_{ji}^{S}$$

The Cauchy (classical) stress tensor, σ_{ij}, and the higher-order stresses, p_i, $\tau_{ijk}^{(1)}$ and m_{ij} are the work-conjugate to the deformation measures ε_{ij}, γ_i, $\eta_{ijk}^{(1)}$ and χ_{ij}^{S}, respectively and are given by constitutive relations as follows

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{mm} + 2\mu\varepsilon_{ij} \tag{7}$$

$$p_i = 2\mu\, l_0^2\gamma_i \tag{8}$$

$$\tau_{ijk}^{(1)} = 2\mu\, l_1^2\eta_{ijk}^{(1)} \tag{9}$$

$$m_{ij} = 2\mu\, l_2^2\chi_{ij}^{S} \tag{10}$$

where λ and μ are Lame's constants, and l_0, l_1, l_2 are additional independent material length scale parameters corresponding to dilatation gradients, deviatoric stretch gradients and rotation gradients, respectively. The parameters for specific materials can be measured by several typical experiments such as micro-bend test, micro-torsion test and specially micro/nano indentation test (Lam et al., 2003; Yang et al., 2002; Fleck and Hutchinson, 1993; McFarland and Colton, 2005; Stolken and Evans, 1998).

An initially flat plate made of homogeneous linearly elastic material with length a, width b and thickness h is shown in Fig. 1. The x–y plane coincides with the undeformed mid-plane occupying the two dimensional domain Ω bounded by the curve Γ which is piecewise smooth. According to the classical plate theory, the displacement field is expressed as (Reddy, 2004)

$$\begin{aligned} u_x(x,y,z) &= -zw_{,x} \\ u_y(x,y,z) &= -zw_{,y} \\ u_z &= w(x,y) \end{aligned} \tag{11}$$

where u_x, u_y, and u_z represent the x, y and z components of the displacement vector, respectively.

Using the displacement field given by Eq. (11), the strain tensor components which are non-zero can be written as

$$\begin{aligned} \varepsilon_{xx} &= -zw_{,xx} \\ \varepsilon_{yy} &= -zw_{,yy} \\ \varepsilon_{xy} &= -zw_{,xy} \end{aligned} \tag{12}$$

Other deformation measures including the dilatation gradient, the deviatoric stretch gradient and the symmetric rotation gradient can be obtained by substituting the displacement field (11) into Eqs. (3)–(5). For keeping brevity, the results are presented in Appendix A. The next step is to calculate the stresses including classical stress and higher-order stresses. After the appropriate replacement of the Lame's constants by modulus of elasticity E and the Poisson's ratio ν in Eq. (7), the non-zero stresses can be given by (Timoshenko and Goodier, 1970)

$$\begin{aligned} \sigma_{xx} &= \frac{E}{1-\nu^2}\left(\varepsilon_{xx}+\nu\varepsilon_{yy}\right) = -\frac{Ez}{1-\nu^2}\left(w_{,xx}+\nu w_{,yy}\right) \\ \sigma_{yy} &= \frac{E}{1-\nu^2}\left(\nu\varepsilon_{xx}+\varepsilon_{yy}\right) = -\frac{Ez}{1-\nu^2}\left(\nu w_{,xx}+w_{,yy}\right) \\ \sigma_{xy} &= \frac{E}{1+\nu}\varepsilon_{xy} = -\frac{Ez}{1+\nu}w_{,xy} \end{aligned} \tag{13}$$

Wang et al. (2011) used three-dimensional general stress–strain relations instead of above two dimensional (2D) equations, which led to different results. Several references can be referred applying the plane stress–strain relations to analysis of classical plate in the category of the higher-order elasticity theories (Tsiatas, 2009; Reddy and Kim, 2012; Ramezani, 2012). Keeping in mind that in the development of the higher-order elasticity theories such as couple stress, modified couple stress and the present modified strain gradient theories, all of the higher-order theories could be reduced to classical elasticity once one or more material length scale parameters are taken to be zero whereas Eq. (24) of the paper (Wang et al., 2011) cannot change to classical Kirchhoff's plate biharmonic equation by imposing $l_0 = l_1 = l_2 = 0$. It can be evidence that 2D above stress–strain relation must be used to achieve proper plate equation. Higher-order stresses can be calculated in a similar manner by utilizing Eqs. (8)–(10) and results are presented in Appendix A. Thus the first variation of the strain energy density given by Eq. (1) can be rewritten as

$$\begin{aligned} \delta U = \int_V \Big(& \sigma_{xx}\delta\varepsilon_{xx} + 2\sigma_{xy}\delta\varepsilon_{xy} + \sigma_{yy}\delta\varepsilon_{yy} + p_x\delta\gamma_x + p_y\delta\gamma_y \\ & + p_z\delta\gamma_z + \tau_{xxx}^{(1)}\delta\eta_{xxx}^{(1)} + 3\tau_{xxy}^{(1)}\delta\eta_{xxy}^{(1)} + 3\tau_{xxz}^{(1)}\delta\eta_{xxz}^{(1)} \\ & + 3\tau_{xyy}^{(1)}\delta\eta_{xyy}^{(1)} + \tau_{yyy}^{(1)}\delta\eta_{yyy}^{(1)} + 3\tau_{yyz}^{(1)}\delta\eta_{yyz}^{(1)} + 3\tau_{xzz}^{(1)}\delta\eta_{xzz}^{(1)} \\ & + 3\tau_{yzz}^{(1)}\delta\eta_{yzz}^{(1)} + \tau_{zzz}^{(1)}\delta\eta_{zzz}^{(1)} + 6\tau_{xyz}^{(1)}\delta\eta_{xyz}^{(1)} + m_{xx}\delta\chi_{xx}^{S} \\ & + 2m_{xy}\delta\chi_{xy}^{S} + m_{yy}\delta\chi_{yy}^{S}\Big)\,\mathrm{d}V \\ = \int_\Omega \big(& M_{xx}\delta w_{,xx} + M_{xy}\delta w_{,xy} + M_{yy}\delta w_{,yy} + N_{xxx}\delta w_{,xxx} \\ & + N_{xxy}\delta w_{,xxy} + N_{xyy}\delta w_{,xyy} + N_{yyy}\delta w_{,yyy}\big)\,\mathrm{d}\Omega \end{aligned} \tag{14}$$

in which new parameters are defined as

$$M_{xx} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(-p_z - 4\tau_{xxz}^{(1)} + \tau_{yyz}^{(1)} + \tau_{zzz}^{(1)} - m_{xy}\right)\mathrm{d}z - \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xx}z\,\mathrm{d}z$$

$$M_{xy} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(-10\tau_{xyz}^{(1)} + m_{xx} - m_{yy}\right)\mathrm{d}z - 2\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xy}z\,\mathrm{d}z$$

$$M_{yy} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(-p_z + \tau_{xxz}^{(1)} - 4\tau_{yyz}^{(1)} + \tau_{zzz}^{(1)} + m_{xy}\right)\mathrm{d}z - \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{yy}z\,\mathrm{d}z$$

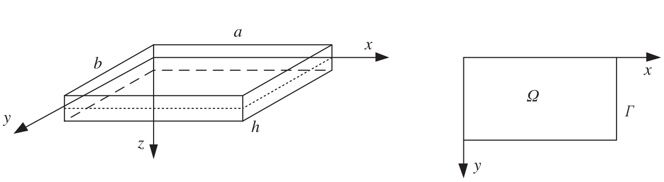

Fig. 1. Geometry of the plate.

$$N_{xxx} = \frac{1}{5}\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(-5p_x - 2\tau_{xxx}^{(1)} + 3\tau_{xyy}^{(1)} + 3\tau_{xzz}^{(1)}\right)z\,dz$$

$$N_{xxy} = \frac{1}{5}\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(-5p_y - 12\tau_{xxy}^{(1)} + 3\tau_{yyy}^{(1)} + 3\tau_{yzz}^{(1)}\right)z\,dz$$

$$N_{xyy} = \frac{1}{5}\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(-5p_x - 12\tau_{xyy}^{(1)} + 3\tau_{xxx}^{(1)} + 3\tau_{xzz}^{(1)}\right)z\,dz$$

$$N_{yyy} = \frac{1}{5}\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(-5p_y - 2\tau_{yyy}^{(1)} + 3\tau_{xxy}^{(1)} + 3\tau_{yzz}^{(1)}\right)z\,dz \tag{15}$$

Above-mentioned expressions are elucidated in terms of deflection in Appendix A. Using the divergence theorem leads to

$$\begin{aligned}\delta U = &\int_{\Omega}(M_{xx,xx} + M_{xy,xy} + M_{yy,yy} - N_{xxx,xxx} - N_{xxy,xxy} - N_{xyy,xyy} \\ &- N_{yyy,yyy})\,\delta w d\Omega + \oint_{\Gamma}\{(-M_{xx,x} - M_{xy,y} + N_{xxx,xx} \\ &+ N_{xxy,xy})n_x + (-M_{yy,y} + N_{xyy,xy} + N_{yyy,yy})n_y\}\,\delta w ds \\ &+ \oint_{\Gamma}\{(M_{xx} - N_{xxx,x} - N_{xxy,y})n_x + M_{xy}n_y\}\,\delta w_{,x} ds \\ &+ \oint_{\Gamma}(M_{yy} - N_{xyy,x} - N_{yyy,y})\,n_y\delta w_{,y} ds + \oint_{\Gamma}(N_{xxx}n_x \\ &+ N_{xxy}n_y)\,\delta w_{,xx} ds + \oint_{\Gamma}(N_{xyy}n_x + N_{yyy}n_y)\,\delta w_{,yy} ds\end{aligned} \tag{16}$$

The first variations of the work done by external forces including distributed load $q(x,y)$takes the following form

$$\delta W = \int_{\Omega} q\delta w\, d\Omega \tag{17}$$

The minimum potential energy principle is written as

$$\delta\Pi = \delta U - \delta W = 0 \tag{18}$$

By substituting the results for both strain energy density and work done by external forces in minimum potential energy principle and also using fundamental lemma of calculus of variation, the governing differential equation is obtained as

$$M_{xx,xx} + M_{xy,xy} + M_{yy,yy} - N_{xxx,xxx} - N_{xxy,xxy} - N_{xyy,xyy} - N_{yyy,yyy} = q \tag{19}$$

By using Appendix A, One can after lengthy but straightforward manipulations derive the governing differential equation in term of deflection in the following form

$$D\nabla^4 w - K\nabla^6 w = q \tag{20}$$

in which constants D and K are

$$D = \frac{Eh^3}{12(1-\nu^2)} + \mu h\left(2l_0^2 + \frac{8}{15}l_1^2 + l_2^2\right),\quad K = \mu h^3\left(\frac{l_0^2}{6} + \frac{l_1^2}{15}\right) \tag{21}$$

It is observed that Eq. (20) leads to the Kirchhoff plate governing equation using the modified couple stress theory (Tsiatas, 2009) or the classical theory for $l_0 = l_1 = 0$and$l_0 = l_1 = l_2 = 0$, respectively. The boundary conditions are simplified for two special cases

- case (i): simply supported

$$\begin{aligned} &w = 0,\quad \frac{\partial^3 w}{\partial x^3} = 0,\quad D\frac{\partial^2 w}{\partial x^2} - K\frac{\partial^4 w}{\partial x^4} = 0 \qquad x = 0, a \\ &w = 0,\quad \frac{\partial^3 w}{\partial y^3} = 0,\quad D\frac{\partial^2 w}{\partial y^2} - K\frac{\partial^4 w}{\partial y^4} = 0 \qquad y = 0, b\end{aligned} \tag{22}$$

- case (ii): clamped

$$\begin{aligned} &w = 0,\quad \frac{\partial w}{\partial x} = 0,\quad \frac{\partial^3 w}{\partial x^3} = 0 \qquad x = 0, a \\ &w = 0,\quad \frac{\partial w}{\partial y} = 0,\quad \frac{\partial^3 w}{\partial y^3} = 0 \qquad y = 0, b\end{aligned} \tag{23}$$

The above-derived boundary conditions are meaningful resulted by classical material behavior of supports which are different from those obtained by Wang et al. (2011) Indeed, boundary conditions given by Eq. (27) of the paper (Wang et al., 2011) have been simplified incorrectly for simply supported plate referred by Appendix B.

3. Implementation of the EKM

According to the general procedure of extended Kantorovich method, deflection should be considered as multiplication of single term separable functions as

$$w(x,y) = \psi(x)\,\eta(y) \tag{24}$$

in which $\psi(x)$ and $\eta(y)$ are unknown functions of x and y to be determined. The boundary conditions in Eqs. (22) and (23) in terms of separable functions can be written as

- case (i)

$$\psi = 0,\quad \frac{d^3\psi}{dx^3} = 0,\quad D\frac{d^2\psi}{dx^2} - K\frac{d^4\psi}{dx^4} = 0 \qquad x = 0, a \tag{25}$$

$$\eta = 0,\quad \frac{d^3\eta}{dy^3} = 0,\quad D\frac{d^2\eta}{dy^2} - K\frac{d^4\eta}{dy^4} = 0 \qquad y = 0, b \tag{26}$$

- case (ii)

$$\psi = 0,\quad \frac{d\psi}{dx} = 0,\quad \frac{d^3\psi}{dx^3} = 0 \qquad x = 0, a \tag{27}$$

$$\eta = 0,\quad \frac{d\eta}{dy} = 0,\quad \frac{d^3\eta}{dy^3} = 0 \qquad y = 0, b \tag{28}$$

In view of Eq. (24), the governing differenti28 al equation given by Eq. (20) can be rewritten in terms of separable functions as

$$D\left(\eta\frac{d^4\psi}{dx^4}+2\frac{d^2\psi}{dx^2}\frac{d^2\eta}{dy^2}+\psi\frac{d^4\eta}{dy^4}\right) - K\left(\eta\frac{d^6\psi}{dx^6}+3\frac{d^4\psi}{dx^4}\frac{d^2\eta}{dy^2}+3\frac{d^2\psi}{dx^2}\frac{d^4\eta}{dy^4}+\psi\frac{d^6\eta}{dy^6}\right) = q \tag{29}$$

According to the general procedure of the weighted residual methods, Eq. (29) should be multiplied by an appropriate weighting function assumed to be $\eta(y)$ (Aghdam and Falahatgar, 2003). Considering an initial guess for $\eta(y)$ and integrating of Eq. (29) multiplied by weighting function over the length of plate in the y direction yield to the first ODE as

$$-KA_0\frac{d^6\psi}{dx^6}+(DA_0-3KA_1)\frac{d^4\psi}{dx^4}+(2DA_1-3KA_2)\frac{d^2\psi}{dx^2} + (DA_2-KA_3)\psi = C_1 \tag{30}$$

where constants are defined as the following form

$$A_m = \int_0^b \eta\frac{d^{2m}\eta}{dy^{2m}}dy,\quad C_1 = \int_0^b \eta q\,dy,\quad (m = 0,1,2,3) \tag{31}$$

By solving the resulted ordinary differential equation given by Eq. (30) as well as boundary conditions (25) or (27), the first approximation for function $\psi(x)$ can be achieved. Applying a similar manipulation in the x direction, i.e. multiplying each sides of Eq. (29) by the obtained approximation function $\psi(x)$ and integrating the resulted equation over the length of the plate in the x direction, results in the second ODE in term of η as

$$-KB_0\frac{d^6\eta}{dy^6}+(DB_0-3KB_1)\frac{d^4\eta}{dy^4}+(2DB_1-3KB_2)\frac{d^2\eta}{dy^2} + (DB_2-KB_3)\eta = C_2 \tag{32}$$

where constants are defined as

$$B_m = \int_0^a \psi\frac{d^{2m}\psi}{dx^{2m}}dx,\quad C_2 = \int_0^a \psi q\,dx,\quad (m = 0,1,2,3) \tag{33}$$

Again, solving the ODE resulted in Eq. (32) together with the boundary conditions (26) or (28) yields the first approximation for $\eta(y)$ and the first iteration is completed. In second iteration, constants of the Eq. (30) can be obtained using new function $\eta(y)$ and the procedure should be continued until the convergence is achieved.

Closed-form solutions can be presented for both ODEs given by Eqs. (30) and (32) in all iterations. The closed-form solutions for a uniformly distributed load are combinations of homogenous and particular solution as

$$\psi(x)=\sum_{i=1}^{6}G_ie^{\alpha_i x}+\frac{C_1}{DA_2-KA_3},\quad \eta(y)=\sum_{i=1}^{6}H_ie^{\beta_i y}+\frac{C_2}{DB_2-KB_3} \tag{34}$$

in which G_i and $H_i(i = 1, 2, \ldots, 6)$are constants of the integration determined by applying boundary conditions.

4. Results and discussion

Using the procedure outlined in the previous section, highly accurate closed-form solutions are presented for bending of a modified strain gradient plate model. The first step is to assume a function for $\eta(y)$ as an initial guess function. The initial guess is arbitrary and is given by

$$\eta(y) = \sin\left(\frac{\pi y}{b}\right) \tag{35}$$

It should be noted that the weighting function is not needed to satisfy all of boundary conditions (Kerr and Alexander, 1961; Dalaei and Kerr, 1995; Aghdam et al., 1996). Using initial guess in Eq. (35) and the expressions given in Eq. (31), all constants of the Eq. (30) are obtained. Solving resulted ODE in conjunction with related boundary conditions lead to determination of the constants of Eq. (32). As a consequence, the new expression for $\eta(y)$ can be obtained by solving Eq. (32) together with corresponding boundary conditions. The procedure is continued until convergence of the results is achieved.

For illustration purpose, the plate considered here is assumed to be made of epoxy with the following properties: $E = 1.44$ GPa, $v = 0.38$, $l = 17.6$ μm (Lam et al., 2003). Geometric properties of the plate are: $a = b = 50$ h, and the external loading is assumed to be uniformly distributed as $q = 1$ kN/m^2.

Convergence rate of the separable function $\psi(x)$ and $\eta(y)$, from which deflection of plate can be determined, are shown in Figs. 2 and 3. The plate thickness is assumed to be $h = l$. Moreover, the identity of all three material length scale parameters is applied, i.e. $l_0 = l_1 = l_2 = l$. Each figure includes two diagrams for both boundary conditions of the case (i) and (ii). The Figures clearly elucidate that the method converges rapidly such that three to four iterations are enough to get a highly accurate closed-form solution. Figs. 2b and 3b prove that the initial guess function is not required to satisfy boundary conditions. Indeed, despite the fact that the initial guess function does not satisfy the clamped boundary conditions, the first iteration depicts a remarkable satisfaction of the boundary conditions. For problems described by differential equation of sixth or higher-order, such as the presented model, the aforementioned advantages make the method more interesting in comparison to the other numerical methods.

Fig. 4 including two diagrams a and b depicts the deflection profile at the line $y = b/2$ of the plate for three different plate aspect

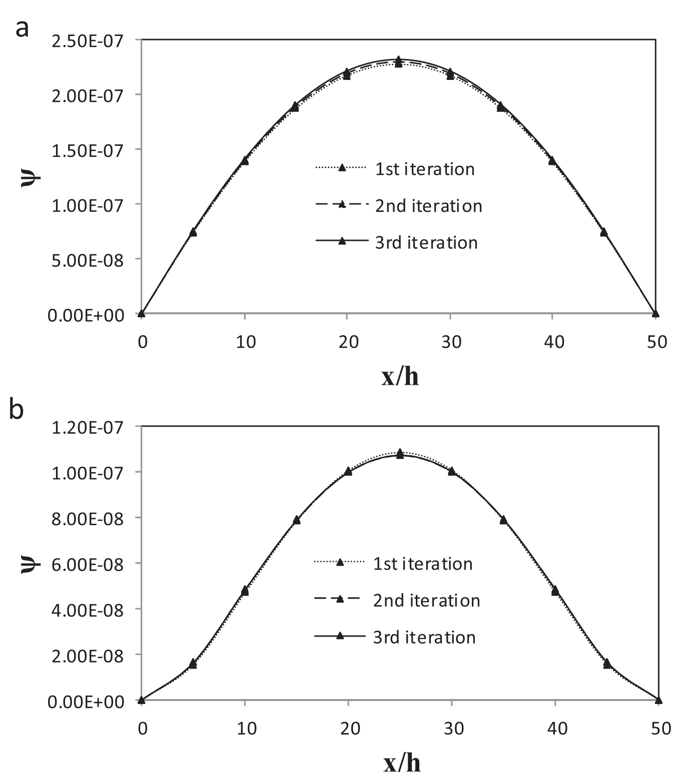

Fig. 2. Convergence rate of separable function $\psi(x)$ for boundary condition (a) case (i) and (b) case (ii).

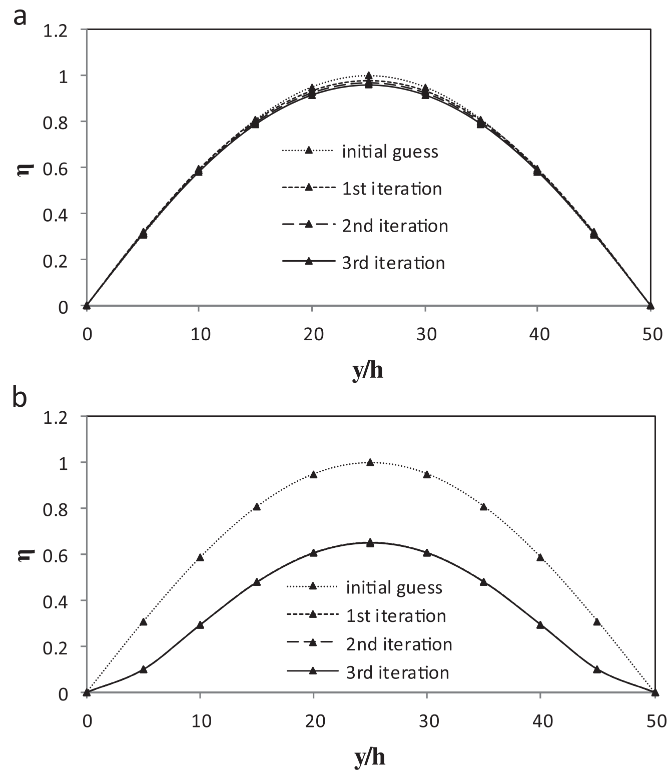

Fig. 3. Convergence rate of separable function $\eta(y)$ for boundary condition (a) case (i) and (b) case (ii).

ratios $r = a/b$. The plate thickness is taken to be $h = l$ in the analysis and all three material length scale parameters are identical, i.e. $l_0 = l_1 = l_2 = l$. Diagram a shows the results for simply supported and diagram b presents those of clamped micro plate. As can be

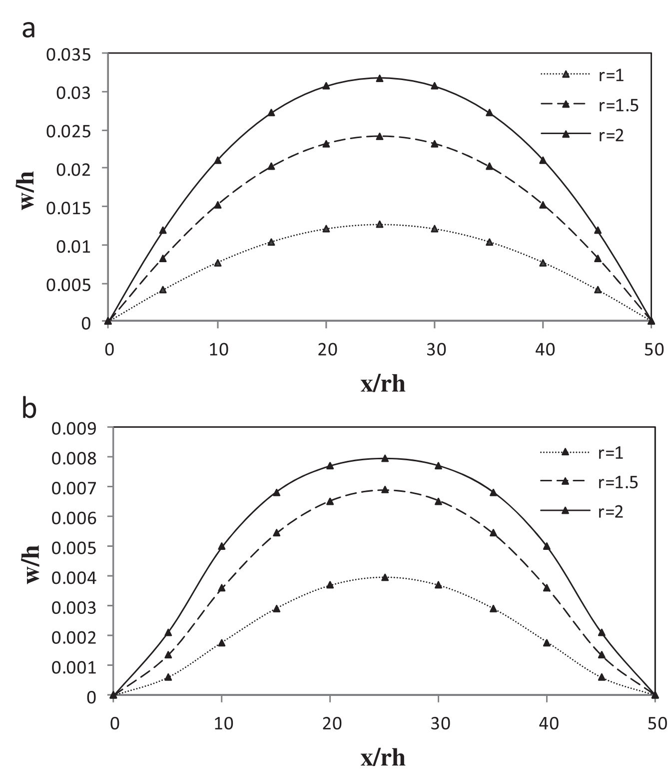

Fig. 4. The effect of aspect ratio of the plate on the deflection at the line $y = b/2$ for boundary condition (a) case (i) and (b) case (ii).

Table 1
The maximum values of the plate deflection.

Boundary condition	w_{max}/h		
	$r = 1$	$r = 1.5$	$r = 2$
Case (i)	0.0127	0.0242	0.0318
Case (ii)	0.004	0.0069	0.008

concluded from the figures, more value of aspect ratio leads to increase the deflection for both boundary conditions of case (i) and (ii). Thus, the same as classical plate theory, the modified strain gradient plate model predicts bending rigidity of the plate decreases while the aspect ratio increases. For a quantitative comparison, the maximum values of the plate deflection are also tabulated in Table 1.

The effect of material length scale parameters is investigated in Fig. 5 including diagram a, b and c considering a simply supported plate. In each diagram, one of the material length scale parameters is changed, while two other parameters are assumed to be

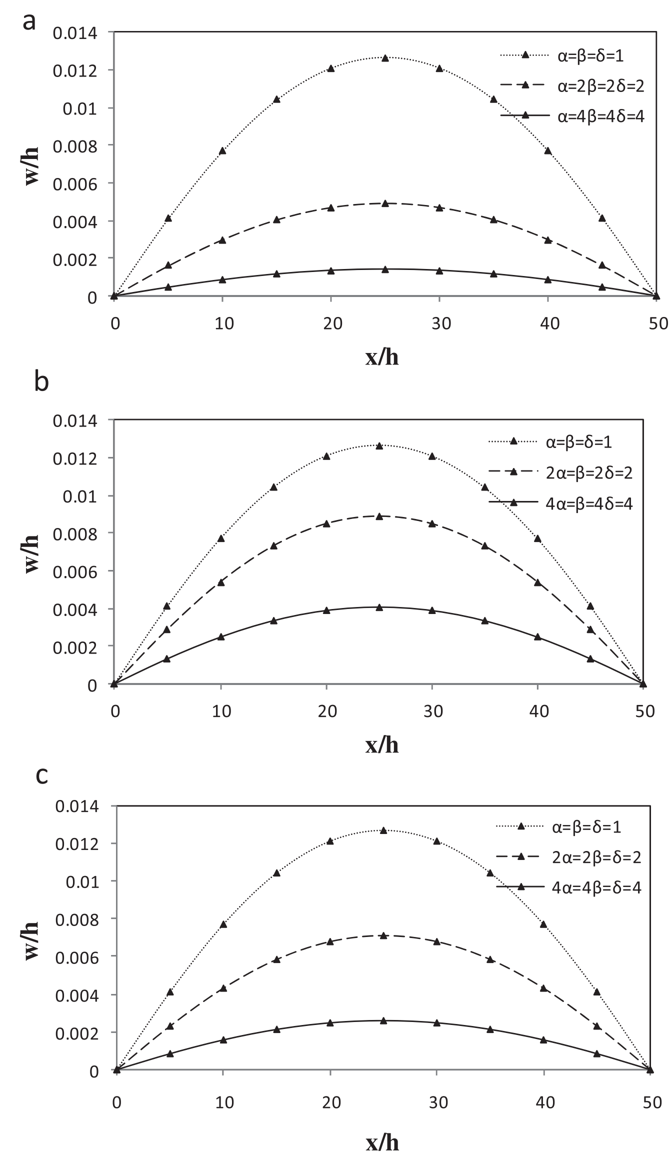

Fig. 5. The effects of the model constants on the deflection profile at the line $y = b/2$ of the micro plate for case (i) boundary condition including the effect of (a) dilatation gradient constant, (b) deviatoric gradient constant and (c) symmetric rotation gradient constant.

constant. The results show the variation of parameters corresponding to the dilatation gradient ($\alpha = l_0/l$), stretch deviatoric gradient ($\beta = l_1/l$) and symmetric rotation gradient ($\delta = l_2/l$), respectively. Similarly, Fig. 6 shows the results for a micro plate with clamped boundary condition. From both Figs. 5 and 6, it should be noted that the parameter associated with the dilatation gradient has the most effect on the deflection, while the parameter associated with the stretch deviatoric gradient has the least effect on the deflection.

In Fig. 7a and b, a comparison is carried out between the modified strain gradient model and its two reduced forms, i.e. the modified couple stress model and the classical model for different values of thickness including both proposed cases of boundary conditions. Fig. 7a shows the results for simply supported boundary condition and those of clamped boundary condition are depicted in Fig. 7b. One can observe from both diagrams of the figure that the deflection of the micro plate predicted by the presented model is smaller than both of the modified couple stress and the classical models. The fact describes that the bending rigidity of the plate increases in the present model in comparison to two other models. Indeed, the modified strain gradient elasticity theory involves the dilatation gradient vector and the deviatoric stretch gradient tensor in addition to the symmetric rotation gradient tensor considered in the modified couple stress theory. Concluding from the Fig. 7a and b, one can distinguish the size-dependency effects of the model. A large difference of deflections related to the three different models appears when the plate thickness h is small and comparable to the material length scale parameters. However, the differences decrease while thickness of the plate increases. The figures also show the differences are negligible when the plate thicknesses are larger than ten times of the length scale parameters. The recent results follow good agreement with the experimental studies which reveal that the size-dependent phenomena exist only for small enough structures (Lam et al., 2003; McFarland and Colton, 2005).

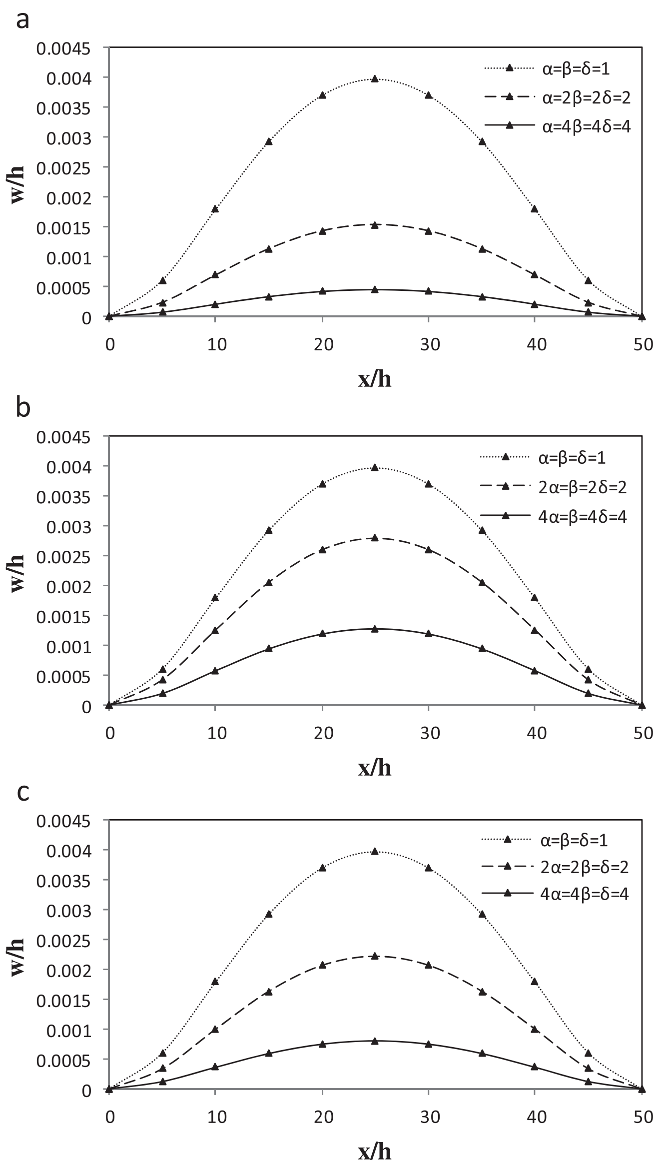

Fig. 6. The effects of the model constants on the deflection profile at the line $y = b/2$ of the micro plate for case (ii) boundary condition including the effect of (a) dilatation gradient constant, (b) deviatoric gradient constant and (c) symmetric rotation gradient constant.

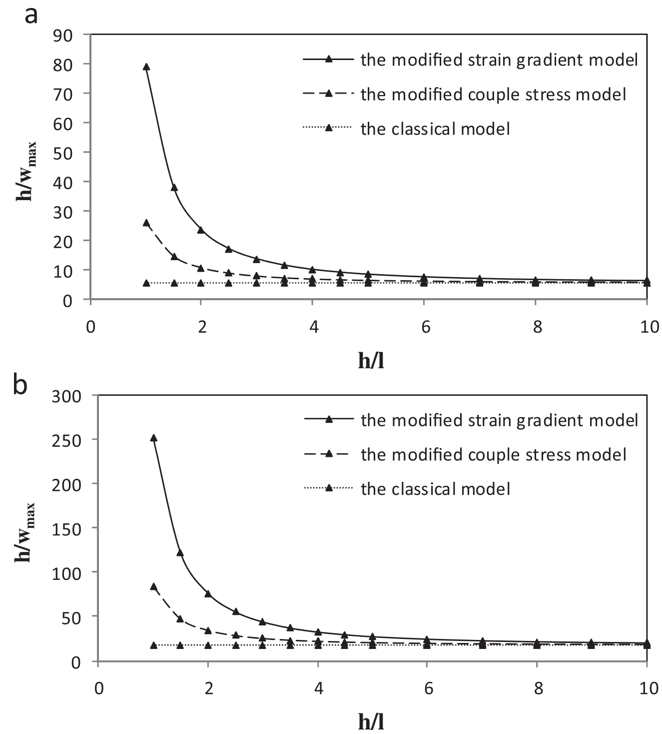

Fig. 7. Comparison of three different models versus thickness of the micro plate for boundary condition (a) case (i) and (b) case (ii).

It worth emphasizing that the micro-plate solution based on the modified strain gradient theory is not available in the literature and the presented model can be reduced to the modified couple stress and the classical theories by setting zero for the l_0, l_1 and l_0, l_1, l_2 length scale parameters, respectively. Therefore, for validation of the presented EKM solution, a comparison is carried out with the results given by Tsiatas (2009) in which the method of fundamental solutions (a boundary-type meshless method) has been used to solve governing equation of the micro plate based on the modified couple stress and classical theories. Geometric and material properties of the plate are (Tsiatas, 2009)

$$\begin{aligned} q &= 1 \qquad \nu = 0.30 \\ a &= b = 1 \quad D = 1 \quad l/h = 0.4 \end{aligned} \tag{36}$$

Fig. 8a depicts the plate deflection for both the modified couple stress and the classical plate model with boundary condition of case (i), and Fig. 8b shows those of the plate with boundary condition of

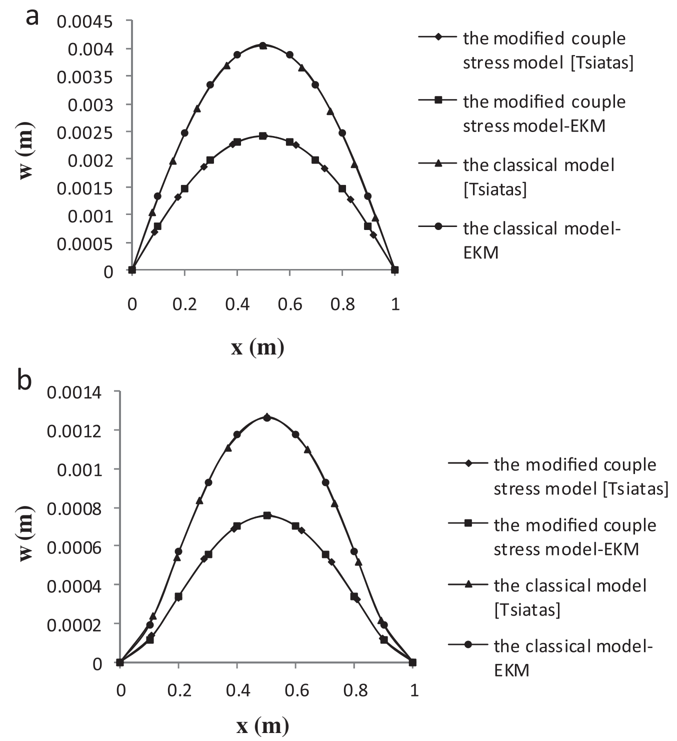

Fig. 8. Comparison of the resulted EKM solution with the meshless method (Tsiatas, 2009) for boundary condition (a) case (i) and (b) case (ii).

case (ii). As one can see, the results show good agreement between the EKM solution and the numerical method.

5. Conclusion

A micro scale Kirchhoff plate formulation is presented based on the modified strain gradient elasticity theory involving three material length scale parameters capturing the size effects. The governing equations in conjunction with the well-proposed form of boundary conditions are obtained using minimum potential energy principle. The capability of extended Kantorovich method is applied in solving resulted PDE in comparison with the other conventional numerical methods. The results are obtained for simply supported and clamped boundary conditions. A comparison of the study is carried out with two other plate models including the modified couple stress plate model and the classical plate model. The numerical results show that the differences between deflections predicted by three models are significant while the plate thickness is small and comparable to the material length scale parameters. However, the differences decrease when plate thickness increases.

Appendix A

The deformation measures are presented below which can be deduced in the following forms.

The dilatation gradient vector:

$$
\begin{aligned}
\gamma_x &= -z\frac{\partial}{\partial x}\left(\nabla^2 w\right)\\
\gamma_y &= -z\frac{\partial}{\partial y}\left(\nabla^2 w\right)\\
\gamma_z &= -\nabla^2 w
\end{aligned}
\tag{37}
$$

The deviatoric stretch gradient tensor:

$$
\begin{aligned}
\eta^{(1)}_{xxx} &= \frac{z}{5}\left(-2\frac{\partial^3 w}{\partial x^3}+3\frac{\partial^3 w}{\partial x\partial y^2}\right)\\
\eta^{(1)}_{xxy} &= \eta^{(1)}_{xyx} = \eta^{(1)}_{yxx} = \frac{z}{5}\left(-4\frac{\partial^3 w}{\partial x^2\partial y}+\frac{\partial^3 w}{\partial y^3}\right)\\
\eta^{(1)}_{xxz} &= \eta^{(1)}_{xzx} = \eta^{(1)}_{zxx} = \frac{1}{15}\left(-4\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)\\
\eta^{(1)}_{xyy} &= \eta^{(1)}_{yxy} = \eta^{(1)}_{yyx} = \frac{z}{5}\left(\frac{\partial^3 w}{\partial x^3}-4\frac{\partial^3 w}{\partial x\partial y^2}\right)\\
\eta^{(1)}_{yyy} &= \frac{z}{5}\left(3\frac{\partial^3 w}{\partial x^2\partial y}-2\frac{\partial^3 w}{\partial y^3}\right)\\
\eta^{(1)}_{yyz} &= \eta^{(1)}_{zyy} = \eta^{(1)}_{yzy} = \frac{1}{15}\left(\frac{\partial^2 w}{\partial x^2}-4\frac{\partial^2 w}{\partial y^2}\right)\\
\eta^{(1)}_{xzz} &= \eta^{(1)}_{zxz} = \eta^{(1)}_{zzx} = \frac{z}{5}\frac{\partial}{\partial x}\left(\nabla^2 w\right)\\
\eta^{(1)}_{yzz} &= \eta^{(1)}_{zyz} = \eta^{(1)}_{zzy} = \frac{z}{5}\frac{\partial}{\partial y}\left(\nabla^2 w\right)\\
\eta^{(1)}_{zzz} &= \frac{1}{5}\nabla^2 w\\
\eta^{(1)}_{xyz} &= \eta^{(1)}_{yzx} = \eta^{(1)}_{zxy} = \eta^{(1)}_{zyx} = \eta^{(1)}_{yxz} = \eta^{(1)}_{xzy} = -\frac{1}{3}\frac{\partial^2 w}{\partial x\partial y}
\end{aligned}
\tag{38}
$$

The symmetric rotation gradient tensor:

$$
\begin{aligned}
\chi^S_{xx} &= \frac{\partial^2 w}{\partial x\partial y}\\
\chi^S_{yy} &= -\frac{\partial^2 w}{\partial x\partial y}\\
\chi^S_{xy} &= \frac{1}{2}\left(\frac{\partial^2 w}{\partial y^2}-\frac{\partial^2 w}{\partial x^2}\right)
\end{aligned}
\tag{39}
$$

Based on the aforementioned deformation measures, one can obtain following results

$$
\begin{aligned}
M_{xx} &= \mu h\left(\frac{h^2}{6(1-\nu)}+2l_0^2+\frac{8}{3}l_1^2+l_2^2\right)w_{,xx}\\
&\quad+\mu h\left(\frac{\nu h^2}{6(1-\nu)}+2l_0^2-\frac{2}{3}l_1^2-l_2^2\right)w_{,yy}\\
M_{xy} &= \mu h\left(\frac{1}{3}h^2+\frac{20}{3}l_1^2+4l_2^2\right)w_{,xy}\\
M_{yy} &= \mu h\left(\frac{\nu h^2}{6(1-\nu)}+2l_0^2-\frac{2}{3}l_1^2+l_2^2\right)w_{,xx}\\
&\quad+\mu h\left(\frac{h^2}{6(1-\nu)}+2l_0^2+\frac{8}{3}l_1^2-l_2^2\right)w_{,yy}\\
N_{xxx} &= \frac{\mu h^3}{6}\left(l_0^2+\frac{2}{5}l_1^2\right)w_{,xxx}+\frac{\mu h^3}{6}\left(l_0^2-\frac{3}{5}l_1^2\right)w_{,xyy}\\
N_{xxy} &= \frac{\mu h^3}{6}\left(l_0^2+\frac{12}{5}l_1^2\right)w_{,xxy}+\frac{\mu h^3}{6}\left(l_0^2-\frac{3}{5}l_1^2\right)w_{,yyy}\\
N_{xyy} &= \frac{\mu h^3}{6}\left(l_0^2-\frac{3}{5}l_1^2\right)w_{,xxx}+\frac{\mu h^3}{6}\left(l_0^2+\frac{12}{5}l_1^2\right)w_{,xyy}\\
N_{yyy} &= \frac{\mu h^3}{6}\left(l_0^2-\frac{3}{5}l_1^2\right)w_{,xxy}+\frac{\mu h^3}{6}\left(l_0^2+\frac{2}{5}l_1^2\right)w_{,yyy}
\end{aligned}
\tag{40}
$$

Appendix B

Boundary conditions extracted by Wang et al. (2011), which are given by Eq. (27) of their paper, are

$$B_{X1}(a,y)\,\delta w(a,y) - B_{X1}(0,y)\,\delta w(0,y) = 0 \tag{41}$$

$$B_{X2}(a,y)\,\delta w_X(a,y) - B_{X2}(0,y)\,\delta w_X(0,y) = 0 \tag{42}$$

$$B_{X3}(a,y)\,\delta w_{XX}(a,y) - B_{X3}(0,y)\,\delta w_{XX}(0,y) = 0 \tag{43}$$

$$B_{Y1}(x,b)\,\delta w(x,b) - B_{Y1}(x,0)\,\delta w(x,0) = 0 \tag{44}$$

$$B_{Y2}(x,b)\,\delta w_Y(x,b) - B_{Y2}(x,0)\,\delta w_Y(x,0) = 0 \tag{45}$$

$$B_{Y3}(x,b)\,\delta w_{YY}(x,b) - B_{Y3}(x,0)\,\delta w_{YY}(x,0) = 0 \tag{46}$$

in which

$$\begin{aligned}
B_{X1}(x,y) &= -2P_1\frac{\partial^3 w}{\partial x^3} - (P_2+2P_3)\frac{\partial^3 w}{\partial x\partial y^2} + 2P_4\frac{\partial^5 w}{\partial x^5} + (2P_5+2P_6)\frac{\partial^5 w}{\partial x^3\partial y^2} + (2P_5+P_6)\frac{\partial^5 w}{\partial x\partial y^4}\\
B_{X2}(x,y) &= 2P_1\frac{\partial^2 w}{\partial x^2} + P_2\frac{\partial^2 w}{\partial y^2} - 2P_4\frac{\partial^4 w}{\partial x^4} - (2P_5+P_6)\frac{\partial^4 w}{\partial x^2\partial y^2} - P_6\frac{\partial^4 w}{\partial y^4}\\
B_{X3}(x,y) &= 2P_4\frac{\partial^3 w}{\partial x^3} + P_6\frac{\partial^3 w}{\partial x\partial y^2}\\
B_{Y1}(x,y) &= -2P_1\frac{\partial^3 w}{\partial y^3} - (P_2+2P_3)\frac{\partial^3 w}{\partial x^2\partial y} + 2P_4\frac{\partial^5 w}{\partial y^5} + (2P_5+2P_6)\frac{\partial^5 w}{\partial x^2\partial y^3} + (2P_5+P_6)\frac{\partial^5 w}{\partial x^4\partial y}\\
B_{Y2}(x,y) &= 2P_1\frac{\partial^2 w}{\partial y^2} + P_2\frac{\partial^2 w}{\partial x^2} - 2P_4\frac{\partial^4 w}{\partial y^4} - (2P_5+P_6)\frac{\partial^4 w}{\partial x^2\partial y^2} - P_6\frac{\partial^4 w}{\partial x^4}\\
B_{Y3}(x,y) &= 2P_4\frac{\partial^3 w}{\partial y^3} + P_6\frac{\partial^3 w}{\partial x^2\partial y}
\end{aligned} \tag{47}$$

According to the fundamental lemma of the calculus of variations used to extract boundary conditions, each term in Eqs. (41)–(46) must be zero because any term is independent from each other. Deflection w is constant and identical to zero at all edges of the simply supported rectangular plate. Therefore, variation of the deflection is also zero at all edges $x = 0, a$ and $y = 0, b$. Consequently, one can obtain the following equality from Eqs. (41) and (44)

$$w = 0, \quad x = 0, a \tag{48}$$

$$w = 0, \quad y = 0, b \tag{49}$$

In Eqs. (42), (43), (45) and (46), the coefficients B_{ij} ($i = X, Y$ and $j = 1, 2, 3$) must vanish because none of the corresponding variation terms equal zero at the edges. The deflection w is constant and zero at all point of the edges $x = 0, a$. Hence, the slope in the y direction, $\partial w/\partial y$ is also zero along the edges $x = 0, a$. Assuming the continuity condition for the deflection functions and all of the partial derivatives, one can interchange the place of derivatives such as

$$\frac{\partial^3 w}{\partial y\partial x^2} = \frac{\partial}{\partial y}\,\frac{\partial^2}{\partial x^2}(w) = \frac{\partial^2}{\partial x^2}\,\frac{\partial}{\partial y}(w) = \frac{\partial^2}{\partial x^2}\left(\frac{\partial w}{\partial y}\right)$$

Consequently, any term contains differentiation with respect to y direction on the boundary $x = 0, a$ vanishes. Therefore, by setting zero value for each term in the Eqs (42) and (43), the boundary conditions on the edges $x = 0, a$ can be simplified as follows

$$2P_1\frac{\partial^2 w}{\partial x^2} - 2P_4\frac{\partial^4 w}{\partial x^4} = 0 \tag{50}$$

$$2P_4\frac{\partial^3 w}{\partial x^3} = 0 \tag{51}$$

In a similar manner, $\partial w/\partial x$ is zero at the edges $y = 0, b$. Accordingly, on the edges $y = 0, b$, Eqs. (45) and (46) lead to

$$2P_1\frac{\partial^2 w}{\partial y^2} - 2P_4\frac{\partial^4 w}{\partial y^4} = 0 \tag{52}$$

$$2P_4\frac{\partial^3 w}{\partial y^3} = 0 \tag{53}$$

Eqs. (48) and (49) as well as Eqs. (50)–(53) represent complete form of the boundary conditions for a simply supported rectangular plate. The set of Eqs. (48)–(53) ensures the required 2×6 boundary conditions. Theses equations are listed below for each edges in the x and y directions, separately

$$\begin{aligned}
&x = 0, a\\
&w = 0, \quad P_1\frac{\partial^2 w}{\partial x^2} - P_4\frac{\partial^4 w}{\partial x^4} = 0, \quad \frac{\partial^3 w}{\partial x^3} = 0\\
&y = 0, b\\
&w = 0, \quad P_1\frac{\partial^2 w}{\partial y^2} - P_4\frac{\partial^4 w}{\partial y^4} = 0, \quad \frac{\partial^3 w}{\partial y^3} = 0
\end{aligned} \tag{54}$$

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