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Adaptive sliding mode control for fuzzy singular systems with time delay and input nonlinearity

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Summary

This paper addresses the sliding mode control problem for a class of uncertain Takagi-Sugeno fuzzy singular systems with state delay and subject to input nonlinearity. Our purpose is focused on designing an adaptive sliding mode controller for such a complex system. First, a new fuzzy integral-type sliding function is designed. Then, an adaptive sliding mode control scheme is established such that the resulting closed-loop system is insensitive to all admissible uncertainties and satisfies the reaching condition. Moreover, delay-dependent sufficient conditions are derived such that the admissibility and the L_2-L_∞ performance requirement of the sliding mode dynamics can be guaranteed in the presence of time delays, external disturbances, and sector nonlinearity input. Finally, the validity and applicability of the proposed theory are illustrated by a numerical example.

KEYWORDS

adaptive sliding mode control, fuzzy singular systems, LMI, time delay

1 | INTRODUCTION

A singular model, also known as a descriptor model, is a mathematical representation that can provide a convenient and natural description of several physical plants. Singular space-state systems are a generalization of standard space-state systems, which are composed of ordinary differential equations, describing the dynamical part of the physical process and algebraic equations to include the interrelationships between the different components in the system.

In studying singular systems, the system regularity and absence of impulses or causality (for discrete systems) must be guaranteed.[1,2] We emphasize that the delays often occur between different components of many practical systems.[3,4] Thus, the study of singular systems with time delays becomes an extremely important topic of control engineering. In view of the generality of singular models and the time-delay phenomenon, many fundamental control problems have gained much research attention and many relevant results have been reported (see, for example, other works[2,5-9] and the references therein).

It becomes increasingly apparent that fuzzy control provides appealing advantages in several applications. As a powerful method to study nonlinear systems, the Takagi-Sugeno (TS) model–based fuzzy control has become a widespread approach to deal with complex nonlinear systems.[10,11] Based on the TS fuzzy model, there has been considerable research work appearing to address the control problem of nonlinear singular systems in the presence of time delays.[12-17]

On a different research front, sliding mode control (SMC) is one of different robust control schemes used to cope with model uncertainties and nonlinearities by taking advantage of the concepts of sliding mode surface design and equivalent control.[18,19]

Sliding mode control, considered as variable structure control, uses a discontinuous control to drive and then constrain the state trajectories to lie within a neighborhood of a specific switching surface on which the system meets the required control specifications. The key feature and advantages of the SMC approach include (1) insensitivity to variations of uncertainties, (2) disturbance rejection capability, and (3) tracking ability. During the past decades, various SMC approaches have been successfully applied for solving many practical control problems.

The integral SMC has been addressed for uncertain systems in the works of Ding et al[20] and Chang[21] and for stochastic systems in the works of Gao et al[22] and Wu and Ho.[23] Recently, the second-order integral SMC approach has been investigated for uncertain systems with control input delay in the work of Zhang et al.[24] The problem of SMC of the TS fuzzy singular system with time-delay was considered in our other work[25] and in the work of Wang and Fei[26] with the assumption that the local input matrices are constant, ie, $B_i = B$.

On the other hand, most of the proposed approaches for SMC are based on the assumption that the systems under consideration are characterized by linear inputs. However, in practice, due to the physical limitation, the control input seems to have a nonlinear character such as sectors, saturation, and deadzone. Therefore, the input nonlinearities can deteriorate the systems' performances and their effects must be taken into account in analyzing and designing any control scheme. Recently, attention has been paid to input nonlinearity,[27-31] but few works have been undertaken for singular systems.

Enlightened by the aforementioned reasons, it is necessary to develop a new robust control strategy to deal with nonlinear singular systems where the real factors such as time delay, input nonlinearity, and external disturbance are unavoidable and often disrupt the desired performances.

In this paper, the TS fuzzy approach is adopted to investigate the SMC problem for a class nonlinear singular systems in the presence of the aforementioned factors. Precisely, the main contributions can be summarized as follows:

- consideration of a nonrestrictive TS fuzzy singular system where the local input matrices are different and the input is nonlinear;
- proposition and construction of a new sliding function and establishing sufficient conditions to ensure the robust admissibility and the L_2-L_∞ performance of the corresponding sliding mode dynamics by means of the feasibility of a convex optimization problem;
- design of a sliding mode controller for the reaching motion such that trajectories of the resulting closed-loop system can be driven onto a prescribed sliding surface and maintained there for all subsequent times.

The remainder of this paper is organized as follows. In Section 2, system description and preliminaries are presented. The main results are developed in Section 3. In Section 4, a numerical example is provided to show the effectiveness of the proposed scheme, and the conclusions are drawn in Section 5.

Notation. The notations in this paper are quite standard except where otherwise stated. The superscript T stands for matrix transposition; $X \in \mathbb{R}^n$ denotes the n-dimensional Euclidean space, whereas $X \in \mathbb{R}^{n\times m}$ refers to the set of all $n \times m$ real matrices; $X > 0$ (respectively, $X \geq 0$) means that the matrix X is real symmetric positive definite (respectively, positive semidefinite); $l_2[0, \infty)$ is the space of square summable vectors; $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the maximum and minimum eigenvalues of a matrix A, respectively; I and 0 represent the identity matrix and a zero matrix with appropriate dimension, respectively; diag$\cdots$ stands for a block-diagonal matrix, sym(X) stands for $X+X^T$; $\|.\|$ denotes the Euclidean norm of a vector and its induced norm of a matrix. In symmetric block matrices or long matrix expressions, we use a star $*$ to represent a term that is induced by symmetry. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations. To avoid clutter, in the following, μ_i denotes $\mu_i(\theta)$.

2 | SYSTEM DESCRIPTION AND PRELIMINARIES

In this study, we consider a nonlinear singular system with state delay described by the following TS fuzzy model:

$$R_i : \text{If } \theta_1(t) \text{ is } F_1^i \text{and if } \theta_2(t) \text{ is } F_2^i \cdots \text{if } \theta_s(t) \text{ is} F_s^i, \text{ then}$$

$$\begin{cases} E\dot{x}(t) = A_i(t)x(t) + A_{hi}(t)x(t - h(t)) + B_i(\phi(u(t)) + f_i(t, x(t))) + B_{wi}w(t) \\ z(t) = C_i x(t) + C_{hi} x(t - h(t)) \\ x(t) = \varphi(t), t \in [-h_M, 0], \quad i = 1, 2, \ldots, r, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state and $u(t) = [u_1, u_2, \dots, u_m]^T \in \mathbb{R}^m$ is the control input. $\phi(u) = [\phi_1(u_1), \phi_2(u_2), \dots, \phi_m(u_m)]^T$, where $\phi_l(u_l)$ is a continuous nonlinear function for $l = 1, 2, \dots, m$. $w(t) \in \mathbb{R}^w$ is the external disturbance input, $f_i(t, x(t))$ represents the system nonlinearity and any model uncertainties in the system including external disturbances, $z(t) \in \mathbb{R}^s$ is the controlled output, $F_j^i (j = 1 \dots s)$ are fuzzy sets, $\theta(t) = [\theta_1(t), \dots, \theta_s(t)]$ is the premise variable vector. It is assumed that the premise variables do not depend on the input variables $u(t)$, which is needed to avoid a complicated defuzzification process of fuzzy controllers. Delay $h(t)$ is time varying and satisfies

$$h_m \le h(t) \le h_M, \quad \dot{h}(t) \le h_d, \tag{2}$$

where h_m and h_M are constants representing the bounds of the delay and h_d is a positive constant. $\varphi(t)$ is a compatible vector-valued initial function in $[-h_M, 0]$ representing the initial condition of the system. The system disturbance, $w(t)$, is assumed to belong to $L_2[0, \infty)$, ie, $\int_0^\infty w^T(t)w(t)dt < \infty$. This implies that the disturbance has finite energy. Matrix $E \in \mathbb{R}^{n \times n}$ may be singular with rank $(E) = q < n$. $A_i(t) = A_i + \Delta A_i$ and $A_{hi} = A_{hi} + \Delta A_{hi}$ are time-varying system matrices. Matrices A_i, A_{hi}, B_i, B_{wi}, C_i, and C_{hi} are constant with appropriate dimensions, and B_i is of full column rank.

The following assumptions are made in this study.

Assumption 1.

1. $\Delta A_i(t)$ and $\Delta A_{hi}(t)$ are real matrices representing norm-bounded parameter uncertainties and satisfy

$$\left[\, \Delta A_i(t) \;\; \Delta A_{hi}(t) \,\right] = M_i F(t) \left[\, N_i \;\; N_{hi} \,\right], \tag{3}$$

where M_i, N_i, and N_{hi} are known real constant matrices and $F(t)$ is unknown time-varying matrix function satisfying $F^T(t)F(t) \leqslant I$.

2. Nonlinear function $f_i(t, x(t))$, or the so-called matched uncertainty, satisfies

$$f_i\,(t, x(t)) \le \rho_i \|x(t)\|, \tag{4}$$

where ρ_i is a positive unknown constant.

3. Exogenous signal $w(t)$ is bounded by upper bound $\bar{w}$

$$\|w(t)\| \le \bar{w}. \tag{5}$$

4. Nonlinear input $\phi(u)$ applied to the system satisfies $\phi(0) = 0$ and

$$u^T \phi(u) \ge \alpha u^T u, \tag{6}$$

where α is a positive constant.

5. Matrix pair $[A_i, B_i]$ is controllable for all $i = 1, 2, \dots, r$.

Remark 1. As in our other work[25] and in the work of Wang and Fei,[26] it is assumed that $B_i = B, i = 1, 2, \dots, r$. This restrictive assumption is not considered in this present study.

The overall fuzzy model is inferred as follows:

$$\begin{cases} E\dot{x}(t) = \displaystyle\sum_{i=1}^{r} \mu_i\,(\theta(t))\,\{A_i(t)x(t) + A_{hi}(t)x(t-h(t)) + B_i\,(\phi\,(u(t)) + f_i\,(t, x(t))) + B_{wi}w(t)\} \\ z(t) = \displaystyle\sum_{i=1}^{r} \mu_i\,(\theta(t))\,\{C_i x + C_{hi} x\,(t-h(t))\}\,, \end{cases} \tag{7}$$

where $\mu_i(\theta(t))$ is the normalized membership function defined by

$$\mu_i\,(\theta(t)) = \frac{\displaystyle\prod_{j=1}^{s} F_j^i\left(\theta_j(t)\right)}{\displaystyle\sum_{i=1}^{r}\prod_{j=1}^{s} F_j^i\left(\theta_j(t)\right)}, \quad i = 1, 2, \dots, r$$

and $F_j^i(\theta_j(t))$ represents the membership degrees of $\theta_j(t)$ in fuzzy set F_j^i. Note that normalized membership $\mu_i(\theta(t))$ satisfies

$$\mu_i\,(\theta(t)) \ge 0, \quad i = 1, 2, \dots, r \quad \sum_{i=1}^{r} \mu_i\,(\theta(t)) = 1. \tag{8}$$

Before stating the main results, we need to introduce some definitions and some lemmas, which will be used in the next section.

Consider the following unforced linear singular system with time delay:

$$E\dot{x} = Ax(t) + A_h x(t - h(t)), \quad 0 \leq h(t) \leq h_M x(t) = \varphi(t), t \in [-h_M, 0]. \tag{9}$$

Definition 1. (See the work of Dai[1])

1. System (9) is said to be regular if $\det(sE - A) \neq 0$.
2. System (9) is said to be impulse free if $\deg(\det(sE - A)) = \text{rank}(E)$.
3. System (9) is said to be admissible if it is regular, impulse free, and stable.

Singular time-delay system (9) may have an impulsive solution. However, the regularity and nonimpulse of (E, A) guarantee the existence and uniqueness of an impulse-free solution to (9) on $[0, \infty)$.

Lemma 1. (See the work of Gu et al[4])
For any constant matrix $M > 0$, any scalar h_m and h_M with $0 < h_m < h_M$, and vector function $x(t) : [-h_M, -h_m] \to \mathbb{R}^n$ such that the integrals concerned are well defined, then the following holds:

$$-h_r \int_{t-h_M}^{t-h_m} x^T(s)Mx(s)ds \leq -\int_{t-h_M}^{t-h_m} x^T(s)dsM \int_{t-h_M}^{t-h_m} x(s)ds$$

with $h_r = h_M - h_m$.

Lemma 2. (See the work of Petersen[32])
Let M, N, Δ be matrices with Δ satisfying $\Delta^T\Delta \leq I$ and scalar $\varepsilon > 0$. Then, the following inequality holds:

$$\text{sym}(M\Delta N) \leq \varepsilon MM^T + \varepsilon^{-1}N^TN. \tag{10}$$

3 | MAIN RESULT

In this section we aim to design a sliding mode controller for the aforementioned fuzzy singular system (7). The SMC design involves 2 basic steps. First, we design a fuzzy integral sliding surface as a function of the system state such that the sliding mode dynamics restricted to the surface is admissible and satisfies the L_2-L_∞ performance. The second step is to synthesize a suitable sliding mode controller to globally drive the system state trajectories to the predefined switching surface and maintain it there for all subsequent time.

3.1 | Integral sliding mode surface

Choose the following fuzzy switching function:

$$R_i : \text{If } \theta_1(t) \text{ is } F_1^i \text{ and if } \theta_2(t) \text{ is } F_2^i \cdots \text{if } \theta_s(t) \text{ is } F_s^i, \text{then}$$
$$s_i(t) = \mathbb{G}_i\left(Ex(t) - Ex(0)\right) - \mathbb{G}_i \int_0^t (A_i + B_iK_i)x(\tau) + A_{hi}x\left(\tau - h(\tau)\right)d\tau, \tag{11}$$

where $\mathbb{G}_i \in \mathbb{R}^{m\times n}$ is a constant matrix satisfying $\mathbb{G}_iB_i$ is nonsingular, and $K_i \in \mathbb{R}^{m\times n}$ is a real matrix to be designed.

Then, the overall sliding surface can be described as

$$s(t) = \sum_{i=1}^{r} \mu_i s_i(t). \tag{12}$$

According to the SMC theory, when the system trajectories reach onto the switching surface, we have $\dot{s}_i(t) = 0$. Then, the ith equivalent control law can be obtained as follows:

$$\phi_e(u) = -K_ix(t) - f_i\left(x(t)\right) - (\mathbb{G}_iB_i)^{-1}\mathbb{G}_i\left(\Delta A_ix(t) + \Delta A_{di}(t)x\left(t - h(t)\right) + B_{wi}w(t)\right). \tag{13}$$

Substituting (13) into (1), we obtain the following ith fuzzy sliding mode dynamics:

$$R_i \,:\, \text{If } \theta_1(t) \text{ is } F_1^i \text{ and if } \theta_2(t) \text{ is } F_2^i \cdots \text{ if } \theta_s(t) \text{ is } F_s^i, \text{ then}$$
$$\begin{cases} E\dot{x} = \overline{A}_i(t)x(t) + \overline{A}_{hi}(t)x(t) + \overline{B}_{wi}w(t) \\ z(t) = C_i x(t) + C_{hi}x(t-h(t)), \end{cases} \tag{14}$$

and the overall sliding mode dynamics can be described by

$$\begin{cases} E\dot{x}(t)x = \sum_{i=1}^{r} \mu_i \left\{ \overline{A}_i(t)x(t) + \overline{A}_{hi}(t)x(t) + \overline{B}_{wi}w(t) \right\} \\ z(t) = \sum_{i=1}^{r} \mu_i \left\{ C_i x(t) + C_{hi}x(t-h(t)) \right\}, \end{cases} \tag{15}$$

where

$$\begin{aligned} &\overline{\mathbb{G}}_i = I - B_i(\mathbb{G}_i B_i)^{-1}\mathbb{G}_i, \quad \overline{A}_i(t) = \overline{A}_i + \Delta\bar{A}_i, \quad \overline{A}_i = A_i + B_i K_i, \quad \overline{A}_{hi} = A_{hi} + \Delta\overline{A}_{hi}, \\ &\overline{B}_{wi} = \overline{\mathbb{G}}_i B_{wi}, \quad \overline{M}_i = \overline{\mathbb{G}}_i M_i, \quad \left[\Delta\overline{A}_i \;\; \Delta\overline{A}_{hi}\right] = \overline{M}_i F(t) \left[N_i \;\; N_{hi}\right]. \end{aligned} \tag{16}$$

3.2 | L_2-L_∞ sliding mode dynamics analysis

In this subsection, we will develop a sufficient delay-dependent condition that ensures for sliding mode dynamics (15) to be robustly admissible with L_2-L_∞ performance.

Definition 2. Sliding mode dynamics (15) is said to be admissible with L_2-L_∞ performance if it is admissible and for prescribed positive scalar γ, the following norm is satisfied under a zero initial condition:

$$\|T_{zw}(s)\|_{L_2\text{-}L_\infty} = \sup_{0 \neq w(t) \in L_2} \frac{\|z(t)\|_\infty}{\|w(t)\|_2} < \gamma. \tag{17}$$

Theorem 1. *Let h_m, h_M, and h_d be given positive scalars. Sliding mode dynamics of (15) is admissible with L_2-L_∞ norm bound γ, if there exist matrices R, $P > 0$, $Q_1 > 0$, $Q_2 > 0$, $Q_3 > 0$, $Z_1 > 0$, $Z_2 > 0$, $Z_3 > 0$, and $G_j, (j = 1, \dots, 5)$ and positive scalars γ, ε, and ϵ such that the following inequalities hold:*

$$\Psi_{1i}(E, \overline{A}_i, A_{hi}, \overline{B}_{wi}, X, G_j) = \begin{bmatrix} \Phi_i + \text{sym}(\Phi_1) & h_r X & \Gamma_1 & \epsilon\Gamma_2 \\ * & -h_r Z_3 & 0 & 0 \\ * & * & -\epsilon I & 0 \\ * & * & * & -\epsilon I \end{bmatrix} < 0 \tag{18}$$

$$\Psi_{2i}(E, \overline{A}_i, A_{hi}, \overline{B}_{wi}, Y, G_j) = \begin{bmatrix} \Phi_i + \text{sym}(\Phi_1) & h_r Y & \Gamma_1 & \epsilon\Gamma_2 \\ & -h_r Z_3 & 0 & 0 \\ * & * & -\epsilon I & 0 \\ * & * & * & -\epsilon I \end{bmatrix} < 0 \tag{19}$$

$$\Psi_{3i}(E, C_i, C_{hi}) = \begin{bmatrix} -E^T P & * & C_i^T \\ * & -\varepsilon I & C_{hi}^T \\ * & * & -\gamma I \end{bmatrix} < 0, \tag{20}$$

where

$$\Phi_i = \begin{bmatrix} \Phi_{11i} & \Phi_{12i} & \Phi_{13i} & \Phi_{14i} & \Phi_{15i} & G_1\bar{B}_{wi} \\ * & \Phi_{22i} & \Phi_{23i} & \Phi_{24i} & \Phi_{25i} & G_2\bar{B}_{wi} \\ * & * & \Phi_{33i} & 0 & -G_3 & G_3\bar{B}_{wi} \\ * & * & * & \Phi_{44i} & -G_4 & G_4\bar{B}_{wi} \\ * & * & * & * & \Phi_{55i} & G_5\bar{B}_{wi} \\ * & * & * & * & * & -\gamma I \end{bmatrix}$$

$$\Gamma_1 = \left[\boldsymbol{G}^T \quad 0\right]^T M_i, \quad \boldsymbol{G} = \left[G_1^T \; G_2^T \; G_3^T \; G_4^T \; G_5^T\right]^T,$$

$$\Gamma_2 = \left[N_i \;\; N_{hi} \;\; 0 \;\; 0 \;\; 0 \;\; 0\right]^T$$

$$\begin{aligned}
&\Phi_{11i} = Q_1 + Q_2 + Q_3 + \mathrm{sym}(G_1\overline{A}_i) - E^TZ_1E - E^TZ_2E, && \Phi_{12i} = \overline{A}_i^T G_1^T + G_2A_{hi}\\
&\Phi_{13i} = E^TZ_1E + \overline{A}_i^T G_3^T, && \Phi_{14i} = E^TZ_2E + \overline{A}_i^T G_4^T\\
&\Phi_{15i} = P + SR^T + \overline{A}_i^T G_5^T - G_1, && \Phi_{22i} = -(1-h_d)Q_2 + \mathrm{sym}(G_2A_{hi})\\
&\Phi_{23i} = A_{hi}^T G_3^T, && \Phi_{24i} = A_{hi}^T G_4^T\\
&\Phi_{25i} = -G_2 + A_{hi}^T G_5^T, && \Phi_{33i} = -Q_1 - E^TZ_1E\\
&\Phi_{44i} = -Q_3 - E^TZ_2E, && \Phi_{55i} = -\mathrm{sym}(G_5) + h_m^2Z_1 + h_M^2Z_2 + h_rZ_3.\\
&\Phi_1 = \begin{bmatrix} 0 & XE - YE & YE & -XE & 0 \end{bmatrix}
\end{aligned}$$

Proof. The proof of this theorem is divided into 2 parts. The first one is concerned with the regularity and the impulse-free characterizations, and the second one treats the stability property of system (15).

First, we consider the nominal case of (15) (ie, $\Delta\overline{A}_i(t) = 0$, $\Delta\overline{A}_{hi}(t) = 0$, and $\Delta\overline{B}_{wi}(t) = 0$).

Using the following notation:

$$\mathbb{A} = \sum_{i=1}^{r} \mu i\bar{A}_i \quad \mathbb{A}_h = \sum_{i=1}^{r} \mu_i A_{hi} \quad \mathbb{B}_w = \sum_{i=1}^{r} \mu_i B_{wi} \quad \mathbb{C} = \sum_{i=1}^{r} \mu_i C_i, \quad \mathbb{C}_h = \sum_{i=1}^{r} \mu_i C_{hi}, \tag{21}$$

system (15) can be written as

$$\begin{cases} E\dot{x}(t) = \mathbb{A}\,x(t) + \mathbb{A}_h x(t-h(t)) + \mathbb{B}_w w(t)\\ z(t) = \mathbb{C}(t)x(t) + \mathbb{C}_h x(t-h(t))\\ x(t) = \varphi(t),\ t \in [-h_M, 0]. \end{cases} \tag{22}$$

Since $\mathrm{rank}(E) = q \le n$, there always exist 2 nonsingular matrices $\mathbb{M}$ and $\mathbb{N} \in \mathbb{R}^{n\times n}$ such that

$$\hat{\mathrm{E}} = \mathbb{M}E\mathbb{N} = \begin{bmatrix} I_q & 0\\ 0 & 0\end{bmatrix}. \tag{23}$$

Then, R can be characterized as $R = \mathbb{M}^T \begin{bmatrix} 0\\ \hat{\Phi}\end{bmatrix}$, where $\hat{\Phi} \in \mathbb{R}^{(n-q)\times(n-q)}$ is any nonsingular matrix.

We also define

$$\begin{aligned}
&\hat{A} = \mathbb{M}\mathbb{A}\mathbb{N} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12}\\ \hat{A}_{21} & \hat{A}_{22}\end{bmatrix}, && \hat{S} = \mathbb{N}^T S = \begin{bmatrix} \hat{S}_{11}\\ \hat{S}_{21}\end{bmatrix},\\
&\hat{P} = \mathbb{M}^{-T}P\mathbb{M}^{-1} = \begin{bmatrix} \hat{P}_{11} & \hat{P}_{12}\\ \hat{P}_{21} & \hat{P}_{22}\end{bmatrix}, && \hat{Z}_l = \mathbb{M}^{-T}Z_l\mathbb{M}^{-1} = \begin{bmatrix} \hat{Z}_{11l} & \hat{Z}_{12l}\\ \hat{Z}_{21l} & \hat{Z}_{22l}\end{bmatrix}, \quad l = 1, 2.
\end{aligned} \tag{24}$$

It follows from (18) that

$$\begin{bmatrix} \phi_{11i} & \phi_{12i}\\ * & \phi_{22i}\end{bmatrix} < 0, \tag{25}$$

where

$$\begin{aligned}
\phi_{11i} &= \mathrm{sym}(G_1\overline{A}_i) - E^TZ_1E - E^TZ_2E\\
\phi_{12i} &= P + SR^T + \overline{A}_i^T G_5^T - G_1\\
\phi_{22i} &= -\,\mathrm{sym}(G_5).
\end{aligned}$$

Premultiplying and postmultiplying (25) by $[\ I\ \ \overline{A}_i^T\ ]$ and its transpose, respectively, we obtain

$$\mathrm{sym}\left(SR^T\overline{A}_i + P\overline{A}_i\right) - E^TZ_1E - E^TZ_2E < 0. \tag{26}$$

Since $\mu_i(\theta) \ge 0$ and $\sum_{i=1}^{r} \mu_i(\theta) = 1$, it yields

$$\mathrm{sym}\left(SR^T\mathbb{A} + P\mathbb{A}\right) - E^TZ_1E - E^TZ_2E < 0. \tag{27}$$

Premultiplying and postmultiplying (27) by $\mathbb{N}^T$ and $\mathbb{N}$, respectively, and then using expressions (23) and (24) yield

$$\mathrm{sym}\left(\hat{S}_{21}\bar{\Phi}^T\hat{A}_{22}\right) < 0, \tag{28}$$

and $\overline{A}_{22}$ is thus nonsingular. Hence, according to Definition 1, singular time-delay system (22) is regular and impulse free for any time delay $h(t)$ satisfying (2).

In the following, we will prove that system (22) is asymptotically stable. For this purpose, we construct a candidate Lyapunov functional as

$$\begin{aligned}
\mathbf{V}(x_t) &= \mathbf{V}_1(x_t) + \mathbf{V}_2(x_t) + \mathbf{V}_3(x_t)\\
\mathbf{V}_1(x_t) &= x^t(t)P^TEx(t)\\
\mathbf{V}_2(x_t) &= \int_{t-h_m}^{t} x^T(s)Q_1x(s)ds + \int_{t-h(t)}^{t} x^T(s)Q_2x(s)ds + \int_{t-h_M}^{x_t} x^T(s)Q_3x(s)ds\\
\mathbf{V}_3(x_t) &= h_m\int_{-h_m}^{0}\int_{t+\theta}^{t}\dot{x}^T(s)E^TZ_1E\dot{x}(s)dsd\theta + h_M\int_{-h_M}^{0}\int_{t+\theta}^{t}\dot{x}^T(s)E^TZ_2E\dot{x}(s)dsd\theta\\
&\quad + \int_{-h_M}^{-h_m}\int_{t+\theta}^{t}\dot{x}^T(s)E^TZ_3E\dot{x}(s)dsd\theta.
\end{aligned} \tag{29}$$

Evaluating the derivative of $\mathbf{V}(x_t)$ along the solutions of system (22), it yields

$$\begin{aligned}
\dot{\mathbf{V}}_1(x_t) &= 2x^T(t)P^TE\dot{x}(t)\\
\dot{\mathbf{V}}_2(x_t) &\leq x^T(t)(Q_1+Q_2+Q_3)x(t) - x^T(t-h_m)Q_1x(t-h_m) - (1-h_d)x^T(t-h(t))Q_2x(t-h(t))\\
&\quad - x^T(t-h_M)Q_3x(t-h_M)\\
\dot{\mathbf{V}}_3(x_t) &= h_m^2\dot{x}^T(t)E^TZ_1E\dot{x}(t) + h_M^2\dot{x}^T(t)E^TZ_2E\dot{x}(t) + h_r\dot{x}^T(t)E^TZ_3E\dot{x}(t) - h_m\int_{t-h_m}^{t}\dot{x}^T(s)E^TZ_1E\dot{x}(s)ds\\
&\quad - h_M\int_{t-h_M}^{t}\dot{x}^T(s)E^TZ_2E\dot{x}(s)ds - \int_{t-h_M}^{t-h_m}\dot{x}^T(s)E^TZ_3E\dot{x}(s)ds.
\end{aligned} \tag{30}$$

From Lemma 1, one can obtain

$$\begin{aligned}
&- h_m\int_{t-h_m}^{t}\dot{x}^T(s)E^TZ_1E\dot{x}(s)ds \leq -\left[x(t)-x(t-h_m)\right]^TE^TZ_1E\left[x(t)-x(t-h_m)\right]\\
&- h_m\int_{t-h_M}^{t}\dot{x}^T(s)E^TZ_2E\dot{x}(s)ds \leq -\left[x(t)-x(t-h_M)^T\right]E^TZ_2E\left[x(t)-x(t-h_M)\right]
\end{aligned} \tag{31}$$

and

$$-\int_{t-h_M}^{t-h_m}\dot{x}^T(s)E^TZ_3E\dot{x}(s)ds = -\int_{t-h_M}^{t-h(t)}\dot{x}^T(s)E^TZ_3E\dot{x}(s)ds - \int_{t-h(t)}^{t-h_m}\dot{x}^T(s)E^TZ_3E\dot{x}(s)ds. \tag{32}$$

Defining $\xi(t) = \left[x^T(t)\ x^T(t-h(t))\ x^T(t-h_m)\ x^T(t-h_M)\ \dot{x}^T(t)E^T\right]^T$, for any appropriately dimensioned matrix X, the following inequality holds:

$$\int_{t-h_M}^{t-h(t)}\begin{bmatrix}\xi(t)\\ E\dot{x}(s)\end{bmatrix}^T\begin{bmatrix}XZ_3^{-1}X^T & X\\ * & Z_3\end{bmatrix}\begin{bmatrix}\xi(t)\\ E\dot{x}(s)\end{bmatrix}ds \geq 0. \tag{33}$$

Then, it is easy to verify

$$-\int_{t-h_M}^{t-h(t)}\dot{x}^T(s)E^TZ_3E\dot{x}(s)ds \leq (h_M-h(t))\xi^T(t)XZ_3^{-1}X^T\xi(t) + 2\xi^T(t)XE(x(t-h(t))-x(t-h_M)). \tag{34}$$

Similarly, for any matrix Y, we obtain

$$-\int_{t-h(t)}^{t-h_m}\dot{x}^T(s)E^TZ_3E\dot{x}(s)ds \leq (h(t)-h_m)\xi^T(t)YZ_3^{-1}Y^T\xi(t) + 2\xi^T(t)YE(x(t-h_m)-x(t-h(t))). \tag{35}$$

From (34)-(35), it can be seen that

$$-\int_{t-h_M}^{t-h_m} \dot{x}^T(s)E^T Z_3 E\dot{x}(s)ds \leq \xi^T(t)\left\{\rho(t)h_r XZ_3^{-1}X^T(1-\rho(t))h_r YZ_3^{-1}Y^T + 2\left[0 \ XE-YE \ YE \ -XE \ 0\right]\right\}\xi(t), \tag{36}$$

where $\rho(t) = \frac{h_M - h(t)}{h_r}$.

From (22), the following equation holds for any matrices $G_j, j = 1, 2, \dots, 5$ with the appropriate dimensions

$$2\xi^T(t)\mathbf{G}[-E\dot{x}(t) + \mathbb{A}x(t) + \mathbb{A}_h x(t-h(t))] = 0. \tag{37}$$

Besides, it is clear that

$$2x^T(t)SR^T E\dot{x}(t) = 0. \tag{38}$$

Considering (30)-(38), we obtain

$$\dot{\mathbf{V}}(x_t) \leq \xi^T(t)\sum_{i=1}^{r}\mu_i\left(\rho(t)\bar{\Psi}_{1i} + (1-\rho(t))\bar{\Psi}_{2i}\right)\xi(t), \tag{39}$$

where

$$\overline{\Psi}_{1i} = \overline{\Phi}_i + \mathrm{sym}(\Phi_1) + h_r XZ_3^{-1}X^T \tag{40}$$

$$\overline{\Psi}_{2i} = \overline{\Phi}_i + \mathrm{sym}(\Phi_1) + h_r YZ_3^{-1}Y^T \tag{41}$$

and

$$\overline{\Phi}_i = \begin{bmatrix} \Phi_{11i} & \Phi_{12i} & \Phi_{13i} & \Phi_{14i} & \Phi_{15i} \\ * & \Phi_{22i} & \Phi_{23i} & \Phi_{24i} & \Phi_{25i} \\ * & * & \Phi_{33i} & 0 & -G_3 \\ * & * & * & \Phi_{44i} & -G_4 \\ * & * & * & * & \Phi_{55i} \end{bmatrix} \tag{42}$$

since $0 \leq \rho(t) \leq 1$, $\rho(t)\overline{\Psi}_{1i} + (1-\rho(t))\overline{\Psi}_{2i}$ is a convex combination of $\overline{\Psi}_{1i}$ and $\overline{\Psi}_{2i}$. If (18)-(19) are satisfied, then by applying the Schur complement, we can verify and obtain that $\rho(t)\overline{\Psi}_{1i} + (1-\rho(t))\overline{\Psi}_{2i} < 0$. Hence, $\dot{\mathbf{V}}(x_t) \leq -\lambda\|\xi(t)\|^2$, which implies that nominal singular system (22) with $w(t) = 0$ is asymptotically stable. Let us now prove that the system has the L_2-L_∞ performance. For this purpose, consider the following performance index:

$$J_0 = \mathbf{V}(x_t) - \gamma\int_0^t w^T(s)w(s)ds, \tag{43}$$

where $\mathbf{V}(x_t)$ is defined as in (29). For any nonzero $w(s) \in L_2$, $t > 0$ and zero initial state condition $\varphi(t) = 0, t \in [-h_M, 0]$, we have

$$J_0 = \mathbf{V}(x_t) - \mathbf{V}(0) - \gamma\int_0^t w^T(s)w(s)ds = \int_0^t \dot{\mathbf{V}}(x_s) - \gamma w^T(s)w(s)ds. \tag{44}$$

Define $\zeta(t) = \left[\xi^T(t) \ w^T(t)\right]^T$. The following null equation holds:

$$2\zeta^T[\mathbf{G}^T \quad 0]^T \times [-E\dot{x} + \mathbb{A}x(t) + \mathbb{A}_h x(t-h(t)) + \mathbb{B}_w w(t)] = 0. \tag{45}$$

By following the same procedure as used above, we can verify that

$$\dot{\mathbf{V}}(x_t) - \gamma w^T(t)w(t) \leq \zeta^T(t)\sum_{i=1}^{r}\mu_i\left(\rho(t)\hat{\Psi}_{1i} + (1-\rho(t))\hat{\Psi}_{2i}\right)\zeta(t), \tag{46}$$

where $\hat{\Psi}_{1i} = \Phi_i + \mathrm{sym}(\Phi_1) + h_r XZ_3^{-1}X^T$ and $\hat{\Psi}_{1i} = \Phi_i + \mathrm{sym}(\Phi_1) + h_r YZ_3^{-1}Y^T$. Using the Schur complement equivalence of (18) and (19), we can verify that

$$\dot{\mathbf{V}}(x_t) - \gamma w^T(t)w(t) < 0. \tag{47}$$

Thus, $J_0 < 0$, and therefore, we can obtain the following inequality:

$$x^T(t)E^T Px(t) \leq \mathbf{V}(x_t) < \gamma\int_0^t w^T(s)w(s)ds. \tag{48}$$

Furthermore, using the Schur complement equivalence to (20), it yields

$$\begin{bmatrix} C_i^T \\ C_{hi}^T \end{bmatrix}\begin{bmatrix} C_i^T \\ C_{hi}^T \end{bmatrix}^T < \gamma\begin{bmatrix} E^T P & 0 \\ 0 & \varepsilon I \end{bmatrix}. \tag{49}$$

Note that

$$
\begin{aligned}
z^T(t)z(t) &= \sum_{i=1}^{r}\mu_i(\theta)\left\{\begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}^T \begin{bmatrix} C_i^T \\ C_{hi}^T \end{bmatrix} \begin{bmatrix} C_i^T \\ C_{hi}^T \end{bmatrix}^T \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}\right\} \\
&\leq \gamma x^T(t)E^TPx(t) \leq \gamma \mathbf{V}(x_t) \\
&< \gamma^2\int_0^t w^T(s)w(s)ds \leq \gamma^2\int_0^\infty w^T(s)w(s)ds.
\end{aligned} \tag{50}
$$

Taking the maximum value of $\|z(t)\|_\infty^2$, we have $\|z(t)\|_\infty^2 < \gamma^2\|w(t)\|_2^2$ for any $0 \neq w(t) \in L_2$. Consider now the uncertain case. By following the same procedure as used above, it is easy to verify that

$$
\begin{bmatrix} \Phi_i + \text{sym}(\Phi_1) & h_rX \\ * & -h_rZ_3 \end{bmatrix} + \text{sym}\left(\Gamma_1F(t)\Gamma_2^T\right) < 0, \tag{51}
$$

$$
\begin{bmatrix} \Phi_i + \text{sym}(\Phi_1) & h_rY \\ * & -h_rZ_3 \end{bmatrix} + \text{sym}\left(\Gamma_1F(t)\Gamma_2^T\right) < 0. \tag{52}
$$

Then, according to Lemma 2, inequalities (18)-(19) hold using the Schur complement. This completes the proof. □

3.3 | L_2-L_∞ sliding mode dynamics synthesis

In this section, we focus our attention on design gains K_i in (12) such that sliding mode dynamics (15) is robustly admissible with L_2-L_∞ norm bound γ.

Let $x_s(t) = K_ix(t)$.The sliding mode dynamics (15) can be written as

$$
\begin{cases}
\tilde{E}\dot{\tilde{x}} = \sum_{i=1}^{r}\mu i\left\{\tilde{A}_i\tilde{x}(t) + \tilde{A}_{hi}\tilde{x}(t-h(t)) + \tilde{B}_{wi}w(t)\right\} \\
z(t) = \sum_{i=1}^{r}\mu i\left\{\tilde{C}_i\tilde{x}(t) + \tilde{C}_{hi}\tilde{x}(t-h(t)) + \tilde{D}_{wi}w(t)\right\},
\end{cases} \tag{53}
$$

where $\tilde{x}(t) = \left[x^T(t)\ x_s^T(t)\right]^T$ and

$$
\begin{aligned}
&\tilde{E} = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix},\ \tilde{A}_i = \begin{bmatrix} A_i & B_i \\ K_i & -I \end{bmatrix},\ \tilde{A}_{hi} = \begin{bmatrix} A_{hi} & 0 \\ 0 & 0 \end{bmatrix},\ \tilde{B}_{wi} = \begin{bmatrix} \overline{B}_{wi} \\ 0 \end{bmatrix},\ \tilde{C}_i = \begin{bmatrix} C_i & 0 \end{bmatrix}, \\
&\tilde{C}_{hi} = \begin{bmatrix} C_{hi} & 0 \end{bmatrix},\ \tilde{M}_i = \begin{bmatrix} \overline{M}_i^T & 0 \end{bmatrix}^T,\ \tilde{N}_i = \begin{bmatrix} N_i & 0 \end{bmatrix},\ \tilde{N}_{hi} = \begin{bmatrix} N_{hi} & 0 \end{bmatrix}.
\end{aligned} \tag{54}
$$

Theorem 2. *Let $\gamma > 0$, $h_m > 0$, $h_M > 0$, $h_d > 0$, and $\lambda_j, (j = 0, 1, \dots, 5)$ be given scalars. Sliding mode dynamics of (15) is admissible with L_2-L_∞ performance γ, if there exist matrices R, $P > 0$, $Q_1 > 0$, $Q_2 > 0$, $Q_3 > 0$, $Z_1 > 0$, $Z_2 > 0$, $Z_3 > 0$, G_{j11}, $G_{j21}, (j = 1, \dots, 5)$, Y_i, and F_i, $i = 0, 1, \dots, r$ such that the following inequalities hold:*

$$
\Psi_{1i}\left(\tilde{E}, \mathbf{A}_{ji}, \mathbf{A}_{hji}, \mathbf{B}_{wji}, X, \mathbf{G}_{ji}\right) < 0 \tag{55}
$$

$$
\Psi_{2i}\left(\tilde{E}, \mathbf{A}_{ji}, \mathbf{A}_{hji}, \mathbf{B}_{wji}, Y, \mathbf{G}_{ji}\right) < 0 \tag{56}
$$

$$
\Psi_{3i}\left(\tilde{E}, \tilde{C}_i, \tilde{C}_{hi}\right) < 0, \tag{57}
$$

where

$$
\begin{aligned}
&\mathbf{A}_{ji} = \begin{bmatrix} G_{j,11}A_i + \mathbb{I}Y_i & G_{j,11}B_i - \mathbb{I}F_i \\ G_{j,21}A_i + \lambda_iY_i & G_{j,21}B_i - \lambda_iF_i \end{bmatrix},\quad \mathbf{A}_{hji} = \begin{bmatrix} G_{j,11}A_{hi} & 0 \\ G_{j,21}A_{hi} & 0 \end{bmatrix}, \\
&\mathbf{B}_{wji} = \begin{bmatrix} G_{j,11}\overline{B}_{wi} \\ G_{j,21}\overline{B}_{wi} \end{bmatrix},\qquad \mathbf{G}_{ji} = \begin{bmatrix} G_{j,11} & \mathbb{I}F_i \\ G_{j,21} & \lambda_iF_i \end{bmatrix}.
\end{aligned} \tag{58}
$$

Furthermore, $K_i = F_i^{-1}Y_i$

Proof. Under the conditions of Theorem 2, a feasible solution satisfies the condition $\Phi_{55i} = -\mathrm{sym}(\mathbf{G}_{ji}) + h_m^2 Z_1 + h_M^2 Z_2 + h_r Z_3 < 0$. This implies that $\mathbf{G}_{ji}$ is nonsingular, and thus, F_i is also nonsingular. Applying Theorem 1 to system (53) with the following particular structures of matrices $\mathbf{G}_{ji}$:

$$\mathbf{G}_{ji} = \begin{bmatrix} G_{j,11i} & \mathbb{I}F_i \\ G_{j,21i} & \lambda_i F_i \end{bmatrix}, \quad \mathbb{I} = \begin{bmatrix} \mathbb{I}_m \\ 0_{(n-m)\times m} \end{bmatrix}, (j = 1, \ldots, 5, i = 1, 2, \ldots, r), \tag{59}$$

conditions (55)-(57) hold by setting $Y_i = F_i K_i$. This completes the proof. □

Remark 2. In order to obtain the minimum value of L_2-L_∞ performance, the minimum allowed γ satisfying the LMIs in Theorem 2 can be computed by solving the following optimization problem:

$$\text{minimize}\,\gamma \text{ subject to LMIs (55)-(57)}. \tag{60}$$

Remark 3. In (53), we construct an augmented dynamic system with a new state matrix showing a decoupling between control matrices B_i and controller gains K_i. This allows us to avoid the bilinearity problem and to study the general case of fuzzy singular systems with no common control matrices.

3.4 | Adaptive SMC law synthesis

After establishing the appropriate switching surface (12), an adaptive SMC law will be designed to guarantee the reachability of the specified sliding surface $s(t) = 0$ even though uncertainties and input nonlinearity are presented.

To achieve the control objective, the following fuzzy dynamic SMC law is employed:

$$\begin{aligned} &\mathbf{R}_i : \text{If } \theta_1(t) \text{ is } F_1^i \text{ and if } \theta_2(t) \text{ is } F_2^i \cdots \text{if } \theta_s(t) \text{ is } F_s^i, \text{ then} \\ &u(t) = -\hat{\alpha}(t)(\psi i + \chi)\frac{s(t)}{\|s(t)\|} \end{aligned} \tag{61}$$

or, equivalently,

$$u(t) = -\hat{\alpha}(t)(\Omega + \chi)\frac{s(t)}{\|s(t)\|}, \tag{62}$$

where ψ_i is designed as

$$\psi_i = \|\mathbb{G}_i\| \left(\|M_i\|\|N_i\|\|x(t)\| + \|M_i\|\|N_{hi}\|\|x(t-h(t))\| + \|B_{wi}\|\bar{w}(t)\right) + \|K_i x(t)\| + \hat{\rho}(t)\|x(t)\|, \tag{63}$$

$\Omega = \sum_{i=1}^r \mu_i \psi_i$, and $\chi > 0$ is a small constant.

The aforementioned adaptive gains are designed as

$$\dot{\hat{\alpha}}(t) = \kappa_1 \hat{\alpha}^3(t)\Omega\|s(t)\|, \quad \dot{\hat{\rho}}(t) = \kappa_2\|x(t)\|\|s(t)\| \tag{64}$$

with $\hat{\alpha}(0) = \alpha_0$ and $\hat{\rho}(0) = 0$. Note that α_0 is a bounded positive initial value of $\hat{\alpha}(t)$, and κ_1 and κ_2 are adjustable positive constants.

This proposed control scheme will drive the state to reach the sliding surface $s(t) = 0$. This fact is stated in Theorem 3.

Theorem 3. *If the adaptive control input $u(t)$ is designed as (62), with adaptive law (64), then the trajectory of system (7) converges to the sliding surface $s(t) = 0$.*

Proof. Consider the following Lyapunov function:

$$V_s(t) = \frac{1}{2}s^T(t)s(t) + \frac{1}{2\kappa_1}\tilde{\alpha}^2(t) + \frac{1}{2\kappa_2}\tilde{\rho}^2(t), \tag{65}$$

where $\tilde{\alpha}(t) = \hat{\alpha}^{-1}(t) - \alpha$, $\tilde{\rho}(t) = \hat{\rho}(t) - \bar{\rho}$, and $\bar{\rho} = \max_i \rho_i$.

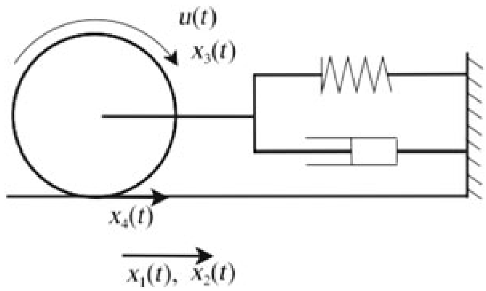

FIGURE 1 Rolling disk

According to (12), we get

$$\dot{s}(t)=\sum_{i=1}^{r}\mu_i\mathbb{G}_i\left\{(\Delta A_i(t)-B_iK_i)x(t)+\Delta A_{hi}(t)x(t-h(t)))+B_{wi}(t)w(t)+B_i\left(\phi(u(t))+f_i(t,x(t))\right)\right\}. \tag{66}$$

Without loss of generality, we can choose $\mathbb{G}_i=B_i^{\dagger}=(B_i^TB_i)^{-1}B_i^T$. Hence, $\mathbb{G}_iB_i$ is nonsingular. By taking the derivative of $V_s(t)$, we get

$$\begin{aligned}
\dot{V}_s(t)&=s^T(t)\dot{s}(t)-\frac{1}{\kappa_1}\tilde{\alpha}(t)\frac{\dot{\hat{\alpha}}(t)}{\hat{\alpha}^2(t)}+\frac{1}{\kappa_2}\tilde{\rho}(t)\dot{\hat{\rho}}(t)\\
&=s^T(t)\sum_{i=1}^{r}\mu_i\mathbb{G}_i\left\{\Delta A_i(t)x(t)+\Delta A_{hi}(t)x(t-h(t)))+B_{wi}(t)w(t)\right\}\\
&\quad+s^T(t)\left(\phi(u(t))+\sum_{i=1}^{r}\mu_i(f_i(t,x(t))-K_ix(t))\right)-\hat{\alpha}(t)\tilde{\alpha}(t)\Omega\|s(t)\|+\tilde{\rho}(t)\|x(t)\|\|s(t)\|\\
&\le\|s(t)\|\sum_{i=1}^{r}\mu_i\left\{\|\mathbb{G}_i\|\left(\|M_i\|\|N_i\|\|x(t)\|+\|M_i\|\|N_{hi}\|\|x(t-h(t))\|+\|B_{wi}\|\bar{w}\right)+\|K_ix(t)\|\right\}\\
&\quad+\bar{\rho}\|x(t)\|\|s(t)\|+\tilde{\rho}\|x(t)\|\|s(t)\|-\hat{\alpha}(t)\tilde{\alpha}(t)\Omega\|s(t)\|+s^T(t)\phi(u(t)).
\end{aligned}\tag{67}$$

Using (62) and Assumption 1.4, it can be derived that

$$u^T(t)\phi(u(t))=-\hat{\alpha}(t)(\Omega+\chi)\frac{s^T(t)}{\|s(t)\|}\phi(u(t))\ge\alpha u^T(t)u(t)=\alpha\hat{\alpha}^2(t)(\Omega+\chi)^2. \tag{68}$$

Since $(\Omega+\chi)>0$, we get

$$s^T(t)\phi(u(t))\le-\alpha\hat{\alpha}(t)(\Omega+\chi)\|s(t)\|. \tag{69}$$

Substituting (69) into (67), we obtain

$$\dot{V}_s(t)=(\Omega-\hat{\alpha}(t)\tilde{\alpha}(t)\Omega-\alpha\hat{\alpha}(t)(\Omega+\chi))\,\|s(t)\|<0,\quad\forall\|s(t)\|\neq0. \tag{70}$$

Noting that $\alpha\hat{\alpha}(t)+\hat{\alpha}(t)\tilde{\alpha}(t)=1$ and $\hat{\alpha}(t)>0$, it is easy to verify that

$$\dot{V}_s(t)<0,\quad\forall t>0. \tag{71}$$

This means the system trajectories converge to the predefined sliding surface and are restricted to the surface for all subsequent time, thereby completing the proof. □

Remark 4. It is well known that discontinuous term $\frac{s(t)}{\|s(t)\|}$ induces the chattering phenomenon. A solution to reduce this phenomenon is to substitute discontinuous function by some continuous and smooth functions as $\frac{s(t)}{\epsilon+\|s(t)\|}$, where ϵ is a small positive scalar value.

4 | A NUMERICAL EXAMPLE

In order to illustrate the efficiency of the approaches proposed in this paper, an example is illustrated in this section.

Example 1. The following nonlinear singular model, represented by a set of differential and algebraic equations, describes a disk rolling on a surface without slipping (see Figure 1). The disk is connected to a fixed wall with a nonlinear spring, a linear damper, and a delayed resonator. K_1 and K_2 are the spring coefficients, both being positive. The damping coefficient of the damper is b, which is also positive. The coefficient of the resonator is C_r. The radius of the disk is r, its inertia is given by J, and the mass of the disk is m. The position of the center of the disk along the surface is given by x_1, whereas x_2 refers to the translational velocity of the same point. The angular velocity of the disk is denoted by x_3. The control input is denoted by u and is a torque applied at the center of the disk. The contact force between the disk and the surface is denoted by x_4. Finally, $w(t)$ is the external disturbance.

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\left(\frac{K_1}{m}x_1 + \frac{K_2}{m}x_1^3\right) - \frac{b}{m}x_2 + \frac{1}{m}x_4 - \frac{C_r}{m}x_2(t-d(t)) + 0.1w(t) \\ \dot{x}_3 = -\frac{r}{J}x_4 + \frac{1}{J}(1 + c_0 cos(x_2))\,(g_u(u(t))u(t) + f(t,x(t)) \\ 0 = x_2 - rx_3 = 0 \end{cases} \tag{72}$$

As in the work of Sjoberg and Glad,[33] this model can be written as

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\left(\frac{K_1}{m}x_1 + \frac{K_2}{m}x_1^3\right) - \frac{b}{m}x_2 + \frac{1}{m}x_4 - \frac{C_r}{m}x_2(t-d(t)) + 0.1w(t) \\ 0 = x_2 - rx_3 \\ 0 = -\left(\frac{K_1}{m}x_1 + \frac{K_2}{m}x_1^3\right) - \frac{b}{m}x_2 + \left(\frac{r^2}{J} + \frac{1}{m}\right)x_4 - \frac{r}{J}(1 + c_0 cos(x_2))\,(g_u(u(t))u(t) + f(t,x(t)), \end{cases} \tag{73}$$

In this example, it assumed that $g_u(u(t)) = (0.4 + 0.3\sin(5u(t))$ and $f(t,x(t)) = (0.1x_1^2 + 0.2\sqrt{|x_2|})e^{-t}$. We select $K_1 = K_2 = K \pm \Delta K = 100 \pm 10\ Nm^{-1}$, $b = 30$ kg/s, $C_r = 10$ kg/s, $c_0 = 0.1$, $m = 40$ kg, $r = 10$ cm, and $J = 3.2$ kgm^{-2}.

For $|x_1(t)| \leq \psi$, the following TS singular fuzzy model is obtained using the sector nonlinearity approach:

$$\begin{cases} E\dot{x} &= \sum_{i=1}^{3} \mu_i(x_1(t))\,\{(A_i + \Delta A_i)x(t) + A_{hi}x(t-h(t)) + B_{wi}w(t) + B_i\,(g_u(u(t))u(t) + f(t,x(t))\} \\ z(t) &= \sum_{i=1}^{3} \mu_i(x_1(t))\,\{C_i x(t) + C_{hi}x(t-h(t))\} \end{cases}$$

$$E = \begin{bmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix},\ A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{K}{m}(3+\psi^2) & -\frac{b}{m} & 0 & \frac{1}{m} \\ 0 & 1 & -r & 0 \\ -\frac{K}{m}(3+\psi^2) & -\frac{b}{m} & 0 & \left(\frac{r^2}{J} + \frac{1}{m}\right) \end{bmatrix},\ A_{hi} = \begin{bmatrix} 0&0&0&0 \\ 0&-\frac{C_r}{m}&0&0 \\ 0&0&0&0 \\ 0&-\frac{C_r}{m}&0&0 \end{bmatrix}$$

$$A_{2,3} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{K}{m} & -\frac{b}{m} & 0 & \frac{1}{m} \\ 0 & 1 & -r & 0 \\ -\frac{K}{m} & -\frac{b}{m} & 0 & \left(\frac{r^2}{J} + \frac{1}{m}\right) \end{bmatrix},\ B_{1,3} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{r^2}{J}(c_0 - 1) \end{bmatrix},\ B_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{r^2}{J}(-c_0 - 1 - \psi^2) \end{bmatrix},\ B_{wi} = \begin{bmatrix} 0 \\ 0.1 \\ 0 \\ 0 \end{bmatrix},$$

where

$$C_{11} = C_{12} = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \end{bmatrix},\ C_{h1} = C_{h2} = \begin{bmatrix} 0 & 0.5 & 0 & 0 \end{bmatrix} \tag{74}$$

$$\mu_1(x_1(t)) = \frac{x_1^2(t)}{\psi^2 + 2},\quad \mu_2(x_1(t)) = \frac{1 + cos(x_2(t))}{\psi^2 + 2},\quad \mu_3(x_1(t)) = \frac{\psi^2 - x_1^2(t) - cos(x_2(t)) + 1}{\psi^2 + 2}.$$

The time-varying delay is given as $h(t) = 0.1 + 0.2e^{-0.5t}$. A straightforward calculation gives $h_m = 0.1$, $h_M = 0.3$, and $h_d = 0.2$. Assume that the uncertain matrices are as follows:

$$M_i = \begin{bmatrix} 0.25 & 0 & 0 & 0 \end{bmatrix}^T,\ N_i = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix},\ N_{hi} = 0 \quad i = 1, 2, 3.$$

Our aim is to design an SMC in the form of (62) such that the sliding mode dynamics is robustly admissible with a guaranteed L_2-L_∞ noise attenuation level. Set

$$\mathbb{G}_i = B_i^{\dagger},\ \lambda_1 = 1.5,\ \lambda_2 = 0.1,\ \lambda_3 = 0,\ \lambda_4 = 0,\ \lambda_5 = 0.05,\ \epsilon = 16.3,\ \varepsilon = 11.$$

Using the YALMIP toolbox with the SeDuMi solver, problem (60) has a feasible solution with the following parameters:

$$P = \begin{bmatrix} 20.0303 & 2.7591 & -36.7289 & -0.9480 & 4.2078 \\ 2.7591 & 9.3207 & -3.7401 & 0.1988 & -0.3779 \\ -36.7289 & -3.7401 & 75.4059 & 1.3429 & -7.3582 \\ -0.9480 & 0.1988 & 1.3429 & 8.4966 & -0.7397 \\ 4.2078 & -0.3779 & -7.3582 & -0.7397 & 8.6754 \end{bmatrix},\ S = \begin{bmatrix} 24.1147 & 12.4407 & -2.1241 \\ -30.4363 & -7.6372 & 1.7454 \\ -24.9101 & 2.8102 & 7.2274 \\ -0.4127 & -8.9164 & 0.7416 \\ 4.1052 & 1.4332 & -7.2607 \end{bmatrix}.$$

The minimum allowed $\gamma^* = 0.054$ and the associate controller gains are

$$\begin{aligned} K_1 &= \begin{bmatrix} -0.2390 & -0.3485 & -0.0840 & -0.0094 \end{bmatrix},\ K_2 = \begin{bmatrix} -0.2163 & -0.3621 & -0.0810 & -0.0106 \end{bmatrix}, \\ K_3 &= \begin{bmatrix} -0.2253 & -0.3671 & -0.0910 & -0.0110 \end{bmatrix}. \end{aligned} \tag{75}$$

The existence of a feasible solution shows that there exists a desire sliding surface in (12) such that the resulting sliding mode dynamics in (15) is admissible with L_2-L_∞ performance.

The remaining task is to design a sliding mode controller such that the system trajectories can be driven onto the predefined sliding surface and maintained there for all subsequent time. For simulation purposes, we take exogenous input $\omega(t) = \cos(2.5t)e^{-0.5t}$ and uncertain matrix function $F(t) = 0.5 + 0.5\sin(2t)$.

With $\chi = 0.75$, $\alpha_0 = 2.5$, $\kappa_1 = 0.1$, and $\kappa_2 = 0.1$, the adaptive SMC law can be designed according to (62)-(64).

To prevent the control signal from chattering, we replace $\frac{s(t)}{\|s(t)\|}$ with $\frac{s(t)}{0.1+\|s(t)\|}$.

The simulation results depicted in Figure 2 show that

- for initial condition $\varphi(t) = [0.1, 0, 0, 0]^T$, $t = -0.2, \ldots, 0$, Figures 2A to 2E depict respectively, the system state trajectories, the control input, the resulting sliding surface, and the adaptive law when the SMC is applied. We observe that the system is stable despite the presence of actuator nonlinearity, parameter uncertainties, and external disturbances.
- Figures 2D and 2E show that the adaptive laws converge to some values depending on initial condition values $\hat{\alpha}(0)$ and $\hat{\rho}(0)$ and on the adaptation gains κ_1 and κ_2. However, we can note that $\hat{\alpha}(t)$ and $\hat{\rho}(t)$ do not necessarily converge to nominal values α and ρ, respectively.
- From Figure 2F, the ratio of $\frac{\|z(t)\|_\infty}{\|w(t)\|_2}$ is less than 0.012 under zero initial condition, which reveals that the L_2-L_∞ disturbance attenuation level is less than required $\gamma^* = 0.054$.
- The proposed scheme can obtain better convergence performance by driving the system trajectories to the specified sliding surface asymptotically instead of to some neighbor of the surface.

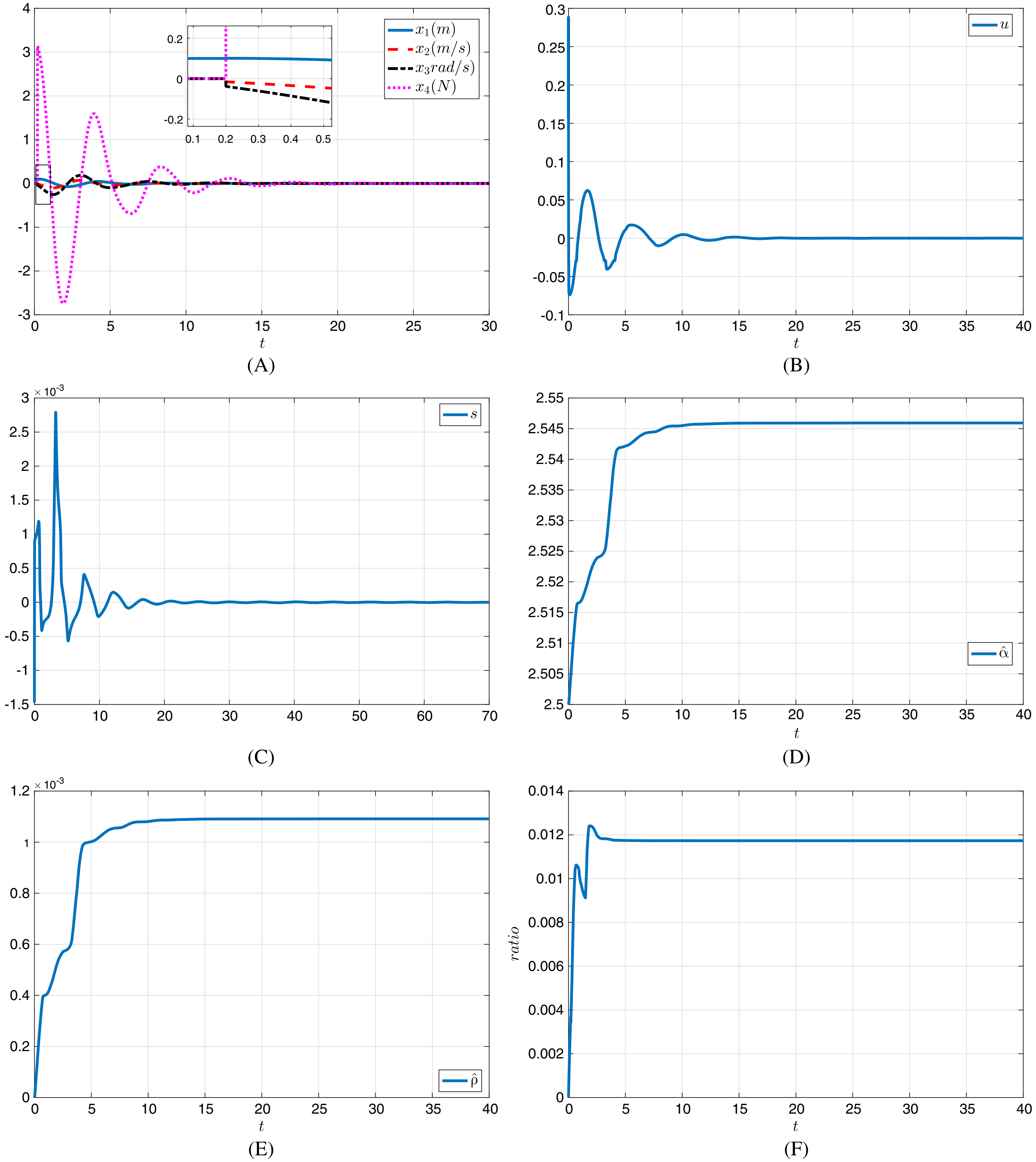

FIGURE 2 Simulation results for rolling disk system. A, State trajectories; B, Input trajectory; C, Surface trajectory; D, Adaptive law; E, Adaptive law; F, Ratio trajectory [Colour figure can be viewed at wileyonlinelibrary.com]

5 | CONCLUSIONS

This paper addresses the problems of SMC for a class of fuzzy singular systems with state delay and nonlinearity input. A fuzzy integral sliding function is proposed and a delay-dependent sufficient condition is derived to guarantee that the sliding mode dynamics is robustly admissible with L_2-L_∞ disturbance rejection performance. Moreover, an adaptive SMC law is designed such that the trajectories of the resulting closed-loop system can be driven onto a prescribed sliding surface and maintained there for all subsequent time. The existence and the effectiveness of theoretical developments have been verified by a numerical example. The proposed controller shows that it has the ability to eliminate the model uncertainties and to reduce the chattering on the sliding surface. It should be emphasized that the computational simplicity of the suggested method can be an another prominent feature of this work.

As future work, we will further investigate the problem of fault estimation and fault-tolerant control for nonlinear singular systems via the SMC scheme.

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