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Optimum design of fuzzy controllers for nonlinear systems using multi-objective particle swarm optimization

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Abstract

In this paper, a multi-objective particle swarm optimization algorithm is used to obtain the Pareto frontiers of the different commensurable and conflicting objective functions for fuzzy controller design. Also, the Lorenz dominance method is used to illustrate the equitable solutions. The nonlinear benchmarks are the inverted pendulum and ball-beam systems. The objective functions for the inverted pendulum system are the normalized angle error of the pendulum and the normalized distance error of the cart; and for the ball-beam system they are the distance error of the ball and the angle error of the beam, which must be minimized simultaneously. The comparison of the obtained results with those in the literature demonstrates the superiority of the results of this work.

Keywords

Particle swarm optimization, fuzzy controller, nonlinear systems, pareto front, Lorenz dominance, equitable solution

1. Introduction

The development of fuzzy controllers for various engineering problems has been a major research activity in recent years (for example, the studies by Harb and Smadi, 2004; Chen, 2011; Bui et al., 2012; Li et al., 2012; and Yeh et al., 2012). In this way, the heuristic parameters of fuzzy controllers have to be determined via an appropriate approach. A very effective way to choose these factors is the use of evolutionary algorithms, such as, for example, the genetic algorithm (GA) and particle swarm optimization (PSO). Belarbi et al. (2005) applied the genetic algorithm to optimum design of Mamdani fuzzy logic controllers. Pourzeynali et al. (2007) implemented the genetic algorithm and fuzzy logic for active control of high rise building structures. Shayeghi et al. (2008) proposed a multi-stage fuzzy controller for solution of the load frequency control which operated under deregulation – the membership functions were also designed automatically by PSO. Larbes et al. (2009) used an optimized fuzzy logic controller to achieve the maximum power point tracking in a photovoltaic system. Marinaki et al. (2010) optimized the fuzzy controller by PSO for vibration suppression of beams. Bingul and Karahan (2011) controlled a two-degrees-of-freedom planer robot by fuzzy logic controller, and PSO was utilized to tune fuzzy parameters. Pan et al. (2011) tuned an optimal fuzzy proportional-integral-derivative controller using GA and two variants of PSO.

PSO was developed via simulation of simplified social systems (Kennedy and Eberhart, 1995) and its robustness has been shown in solving complex optimization problems (Angeline, 1998). This technique can generate high quality solutions with short calculation time and more stable convergence characteristics compared with other evolutionary algorithms (Yoshida et al., 2000). In recent years, researchers have proposed several improved variants of PSO (such as those by

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Janson and Middendorf, 2005; Krohling and Coelho, 2006; Chen and Li, 2007; Chen et al., 2007; Cui et al., 2008; Lu et al., 2010; Wu et al., 2011; and Zhang et al., 2013). Furthermore, numerous methods have been proposed to develop the PSO algorithm for solving multi-objective optimization problems (such as those in Fieldsend and Singh, 2002; Mostaghim and Teich, 2003; Coello and Lechuga, 2004; Heo et al., 2006; Tripathi et al., 2007; Liu et al., 2008; Wang and Yang, 2009; Yen and Leong, 2009; Tsai et al., 2010; Goh et al., 2010; Hernandez-Diaz et al., 2011; Omkar et al., 2012; Urade and Patel, 2012; and Garg and Sharma, 2013). In particular, Mahmoodabadi et al. (2011) and (2012), to increase the ability of the algorithm to find the global minimum and to escape the local minima, combined PSO with novel convergence and divergence operators, and this approach was named CDPSO. In these references, several test functions were used to challenge the capability of the proposed method to solve both single- and multi-objective problems. The results show this algorithm performs very well on the complex benchmarks in terms of solution accuracy and convergence speed. Therefore, in this paper, the periodic multi-objective CDPSO algorithm is employed for the optimum design of the fuzzy controllers for nonlinear systems. Furthermore, to identify the equitable solutions of the Pareto front, the Lorenz dominance is used (Kostreva et al., 2004).

2. Pareto and Lorenz dominance

In most real-world problems, optimization of more than one objective function is required. In such problems the objectives are often in conflict with each other, which means that there is no unique solution for the problem. In other words, there are some optimal solutions for a multi-objective problem that are non-dominated relative to each other, and the designer's task in such situations is to select the best solution according to the requirement. The standard form of a multi-objective optimization problem can be described as follows:

$$\begin{aligned} Minimize : \vec{y} &= f(\vec{x}) = (f_1(\vec{x}), \ldots, f_m(\vec{x})) \\ where : \vec{x} &= (x_1, \ldots, x_n) \in X, \\ \vec{y} &= (y_1, \ldots, y_n) \in Y, \\ g_i(\vec{x}) &= 0, \quad i = 1, 2, \ldots, p \\ h_j(\vec{x}) &\geq 0, \quad j = 1, 2, \ldots,, q \end{aligned} \tag{1}$$

where $\vec{x}$ is the design (decision) vector, X is the space of the design variables, $\vec{y}$ is the objective vector and Y the space of the objective functions. Subject to p equality constraints $g_i(\vec{x}) = 0,\ i = 1, 2, \ldots, p$ and q inequality constraints $h_j(\vec{x}) \geq 0,\ j = 1, 2, \ldots, q$.

As mentioned above, the multi-objective problems do not have a unique optimal solution, but a set of optimal solutions named non-dominated solutions. To describe the concept of optimality, some definitions are given below.

To make it operational, an assumed solution concept specifying what it means to minimize multiple objective functions is required. The solution concepts are defined by the properties of the corresponding preference model. The preference model is completely described by the relation of weak preference (Lopez-de-los-Mosoz and Mesa, 2001). Let the relation of weak preference be denoted by $\preceq$. The corresponding relations of strict preference $\prec$ and indifference $\cong$ are defined as follows.

$$\vec{y}' \prec \vec{y}'' \Leftrightarrow (\vec{y}' \preceq \vec{y}'' \text{ and not } \vec{y}'' \preceq \vec{y}') \tag{2}$$

$$\vec{y}' \cong \vec{y}'' \Leftrightarrow (\vec{y}' \preceq \vec{y}'' \text{ and } \vec{y}'' \preceq \vec{y}') \tag{3}$$

Furthermore, the preference model related to the standard Pareto-optimal solution concept assumes that the preference relation is

- Reflexive, $\vec{y} \preceq \vec{y}$,
- Transitive, $(\vec{y}' \preceq \vec{y}''\ \textit{and}\ \vec{y}'' \preceq \vec{y}''') \Rightarrow \vec{y}' \preceq \vec{y}'''$,
- Strictly monotonic, $\vec{y} - \varepsilon \underset{i}{\vec{e}} \prec \vec{y}$,

where e_i denotes the i-th unit vector in the criterion space. The last assumption states that for each individual objective function less is better (minimization).

The most common multi-objective optimization solution concept is Pareto-optimality. Let $\vec{x}'$ and $\vec{x}''$ be two solutions. The solution $\vec{x}'$ is weakly preferred over $\vec{x}''$, if $f(\vec{x}')$ weakly Pareto dominates $f(\vec{x}'')$: $f(\vec{x}') \preceq_P f(\vec{x}'') \Leftrightarrow f_i(\vec{x}') \leq f_i(\vec{x}''), \quad \forall i \in \{1, \ldots, m\}$. The solution $\vec{x}'$ is preferred over $\vec{x}''$, if $f(\vec{x}')$ Pareto dominates $f(\vec{x}'')$ that means $f(\vec{x}') \prec_P f(\vec{x}'') \Leftrightarrow f(\vec{x}') \preceq_P f(\vec{x}'')$ and $\exists j : f_j(\vec{x}') < f_j(\vec{x}'')$.

The equitable (Lorenz) preference is an enhancement concept of the Pareto preference and assumes that the preference relation is reflexive, transitive, strictly monotonic, impartial, and it holds the principle of transfers.

- Impartiality: while dealing with uniform criteria, we want to focus on the distribution of outcome values while ignoring their ordering. Impartiality makes the distribution of values among the criteria more important than the assignment of values to specific criteria, and therefore models equity among the criteria.

$$\begin{aligned}&(f_{\tau(1)}(\vec{x}),f_{\tau(2)}(\vec{x}),\ldots,f_{\tau(m)}(\vec{x}))\\&\quad\cong(f_1(\vec{x}),f_2(\vec{x}),\ldots,f_m(\vec{x})),\\&\quad\text{for any permutation } \tau \text{ of } \{1,2,\ldots,m\},\ \vec{x}\in X.\end{aligned}$$

- Holding the principle of transfers: a transfer of any small amount from an outcome to any other relatively worse-off outcome results in a more preferred outcome vector.

$$\begin{aligned}f_i(\vec{x})>f_j(\vec{x})\Rightarrow\Big(\Big(&f_1(\vec{x}),\ldots,f_i(\vec{x})-\varepsilon,\ldots,f_j(\vec{x})\\&+\varepsilon,\ldots,f_m(\vec{x})\Big)\prec(f_1(\vec{x}),\ldots,f_m(\vec{x})\Big)\end{aligned}$$

for $0<\varepsilon<f_i(\vec{x})-f_j(\vec{x})$.

Let us consider $f_{(1)}(\vec{x})\geq f_{(2)}(\vec{x})\geq\ldots\geq f_{(m)}(\vec{x})$ as the components of $f=(f_{(1)}(\vec{x}),f_{(2)}(\vec{x}),\ldots,f_{(m)}(\vec{x}))$ sorted by decreasing order. Let $\vec{x}\in X$ be a solution vector. The generalized Lorenz vector associated with $\vec{x}$ is $L(\vec{x})=(l_1,\ldots,l_j,\ldots,l_m)$, where $l_j=\sum_{i=1}^{j}f_{(j)}(\vec{x})$.

Let $\vec{x'},\vec{x''}\in X$ be two solutions. The solution $\vec{x'}$ weakly Lorenz dominates the solution $\vec{x''}$ if:

$\vec{x'}\preceq_L\vec{x''}\Leftrightarrow L(\vec{x'})\preceq_P L(\vec{x''})$. The solution $\vec{x'}$ Lorenz dominates $\vec{x''}$ if: $\vec{x'}\prec_L\vec{x''}\Leftrightarrow L(\vec{x'})\prec_P L(\vec{x''})$.

If a solution $\vec{x}\in X$ is a Lorenz-optimal solution (or simply a Lorenz solution) of a multiple criteria problem, it is also a Pareto-optimal solution.

3. CDPSO algorithm

Mahmoodabadi et al. (2011, 2012) proposed the multi-objective CDPSO algorithm to solve single- and multi-objective problems and demonstrated its success in finding the global optimum of engineering problems in practice. This algorithm is a combination of PSO and two new operators, namely, convergence and divergence operators. In fact, first, a random number $\rho\in[0,1]$ would be allocated for each particle. If a particle has $\rho<CP$ (CP is the convergence probability) then a new particle will be produced by the convergence operator. For each of the particles that are not chosen for the convergence operation, another random number $\vartheta\in[0,1]$ would be allocated. If a particle has $\vartheta<DP$ (DP is the divergence probability), then the divergence operator would generate a new particle. Other particles that are not selected for convergence and divergence operations will be enhanced by the PSO method. This cycle should be repeated until the user-defined stopping criterion is satisfied (Mahmoodabadi et al., 2011, 2012). In the following subsections, a brief review of the PSO, convergence and divergence operators is presented.

3.1. Particle swarm optimization

PSO is one of the recent meta-heuristic optimization algorithms originally introduced by Kennedy and Eberhart (1995) based on the natural flocking and swarming behavior of birds and fish. In essence, the trajectory of each particle is updated according to its own moving experience as well as to that of the best particle in the swarm. In the standard PSO, the movement of the particles is calculated by two equations:

$$\vec{x}_i(t+1)=\vec{x}_i(t)+\vec{v}_i(t+1)\tag{4}$$

$$\begin{aligned}\vec{v}_i(t+1)=W\vec{v}_i(t)&+C_1r_1(\vec{x}_{pbest_i}-\vec{x}_i(t))\\&+C_2r_2(\vec{x}_{gbest}-\vec{x}_i(t))\end{aligned}\tag{5}$$

where $\vec{x}_i(t)$ and $\vec{v}_i(t)$ denotes the position and velocity of particle i at iteration t respectively. C_1 is the cognitive learning factor and represents the attraction that a particle has toward its own success. C_2 is the social learning factor and represents the attraction that a particle has toward the success of the entire swarm. W is the inertia weight that is employed to control the impact of the previous history of velocities on the current velocity of a given particle. $\vec{x}_{pbest_i}$ is the personal best position of the particle i. $\vec{x}_{gbest}$ is the global best position. $r_1,r_2\in[0,1]$ are random values.

3.2. Convergence operator

The convergence mechanism proposed by Mahmoodabadi et al. (2011, 2012) increases the ability of the PSO algorithm to find the global minimum. Assume $\rho\in[0,1]$ to be a random number. If $\rho<CP$ (in this work, $CP=0.1$) then the particle $\vec{x}_i(t)$ moves toward the new particle position $\vec{x}_i(t+1)$ as follows:

If fitness $\vec{x}_i(t)$ is smaller than fitness $\vec{x}_j(t)$ and fitness $\vec{x}_k(t)$ then:

$$\vec{x}_i(t+1)=\vec{x}_{gbest}+\sigma_1\left(\frac{\vec{x}_{gbest}}{\vec{x}_i(t)}\right)\Big(2\vec{x}_i(t)-\vec{x}_j(t)-\vec{x}_k(t)\Big)\tag{6}$$

If fitness $\vec{x}_j(t)$ is smaller than fitness $\vec{x}_i(t)$ and fitness $\vec{x}_k(t)$ then:

$$\vec{x}_i(t+1)=\vec{x}_{gbest}+\sigma_1\left(\frac{\vec{x}_{gbest}}{\vec{x}_i(t)}\right)\Big(2\vec{x}_j(t)-\vec{x}_i(t)-\vec{x}_k(t)\Big)\tag{7}$$

If fitness $\vec{x}_k(t)$ is smaller than fitness $\vec{x}_j(t)$ and fitness $\vec{x}_i(t)$ then:

$$\vec{x}_i(t+1) = \vec{x}_{gbest} + \sigma_1 \left(\frac{\vec{x}_{gbest}}{\vec{x}_i(t)} \right) \left(2\,\vec{x}_k(t) - \vec{x}_j(t) - \vec{x}_i(t) \right) \tag{8}$$

where particles $\vec{x}_j(t)$ and $\vec{x}_k(t)$ are selected from the swarm, randomly. σ_1, σ_2, and σ_3 are random numbers selected from $[0, 1]$ and $\vec{x}_{gbest}$ is the best particle position of the entire swarm. After calculation of equations (3), (4), and (5), the superior between $\vec{x}_i(t)$ and $\vec{x}_i(t+1)$ should be selected (Mahmoodabadi et al., 2011, 2012).

3.3. Divergence operator

In order to prevent falling in to the local minimum points, the divergence mechanism was introduced by Mahmoodabadi et al. (2011, 2012). Suppose $\vartheta \in [0, 1]$ to be a random number. If $\vartheta < DP$, (here, $DP = 0.1$) and particle $\vec{x}_i(t)$ was not enhanced by convergence operator then the following divergence operator is used to generate the new particle $\vec{x}_i(t+1)$.

$$\vec{x}_i(t+1) = Normrand\,(\vec{x}_i(t),\ SD) \tag{9}$$

$Normrand\,(\vec{x}_i(t), SD)$ generates random numbers from the normal distribution with mean $\vec{x}_i(t)$ and standard deviation SD. SD is a positive constant and in this work $SD = \frac{x_{\text{maximum}} - x_{\text{minimum}}}{2}$. x_{maximum} and x_{minimum} are the upper and lower bounds of the search range (Mahmoodabadi et al., 2011, 2012).

4. Periodic multi-objective CDPSO algorithm

The multi-objective optimization problem is defined as the simultaneous optimization of two or more objective functions. In most real world engineering problems, these objectives conflict, and hence, there is no single solution for these problems. Instead, they have a set of optimal solutions commonly referred to as the Pareto optimal set.

Mahmoodabadi et al. (2011, 2012) modified the PSO strategy for solving the multi-objective optimization problems and named their approach periodic multi-objective CDPSO. In the following text, a brief review of the global best selection, inertia weight and learning factors' calculation, and adaptive elimination technique is presented.

4.1. Selection of the global best position

To determine the global best position ($\vec{x}_{gbest}$) of each particle, Mahmoodabadi et al. (2011, 2012) proposed a selection technique based on density measures in the Pareto front set. In this technique, a neighborhood radius $R_{neighborhood}$ is labeled for all non-dominated solutions. Two members of the Pareto front set are neighbors if the Euclidean distance between them is less than $R_{neighborhood}$. Using this definition, the particle with fewer neighbors would be assigned as the global best position. Also, the maximum iteration is divided into several equal periods and in each period this operation would be performed. Moreover, for each particle, a member of the Pareto front set would be assigned to $\vec{x}_{pbest_i}$ by the uniform random method (Mahmoodabadi et al., 2011, 2012).

4.2. Inertia weight and learning factors

Linear formulations for inertia weight and learning factors for multi-objective optimization were proposed by Heo et al. (2006). Furthermore, the method described for the global best position enables us to compute the inertia weight and learning factors in each period as independent of other periods. Therefore, the following equations were suggested to calculate the inertia weight and learning factors in each period (Mahmoodabadi et al., 2011, 2012):

$$W = W_1 - (W_1 - W_2) \times \left(\frac{t}{T} - fix\left(\frac{t-1}{T} \right) \right) \tag{10}$$

$$C_1 = C_{1i} - \left(C_{1i} - C_{1f} \right) \times \left(\frac{t}{T} - fix\left(\frac{t-1}{T} \right) \right) \tag{11}$$

$$C_2 = C_{2i} - \left(C_{2i} - C_{2f} \right) \times \left(\frac{t}{T} - fix\left(\frac{t-1}{T} \right) \right) \tag{12}$$

W_1 and W_2 are the initial and final values of the inertia weight in each period, respectively. C_{1i}, C_{1f}, C_{2i} and C_{2f} are the boundary values of the cognitive and social learning factors in each period, respectively. t is the current iteration number and T is the number of iterations in a period. $fix(\frac{t-1}{T})$ is a function that rounds $(\frac{t-1}{T})$ to the nearest integer in the direction of zero. Figures 1 to 3 illustrate the behavior of the inertia weight and learning factors based on proposed formulations. In these figures, $0.4 \leq W \leq 0.83$, $0.5 \leq C_1 \leq 2.2$, $0.79 \leq C_2 \leq 2.5$, the maximum iteration is 150, and the number of iterations in a period is $T = 7$.

4.3. Control of the archive size

An adaptive elimination technique was employed in the periodic multi-objective CDPSO to delete the crowded members and to create a uniform distribution among the members of the archive (Mahmoodabadi et al.,

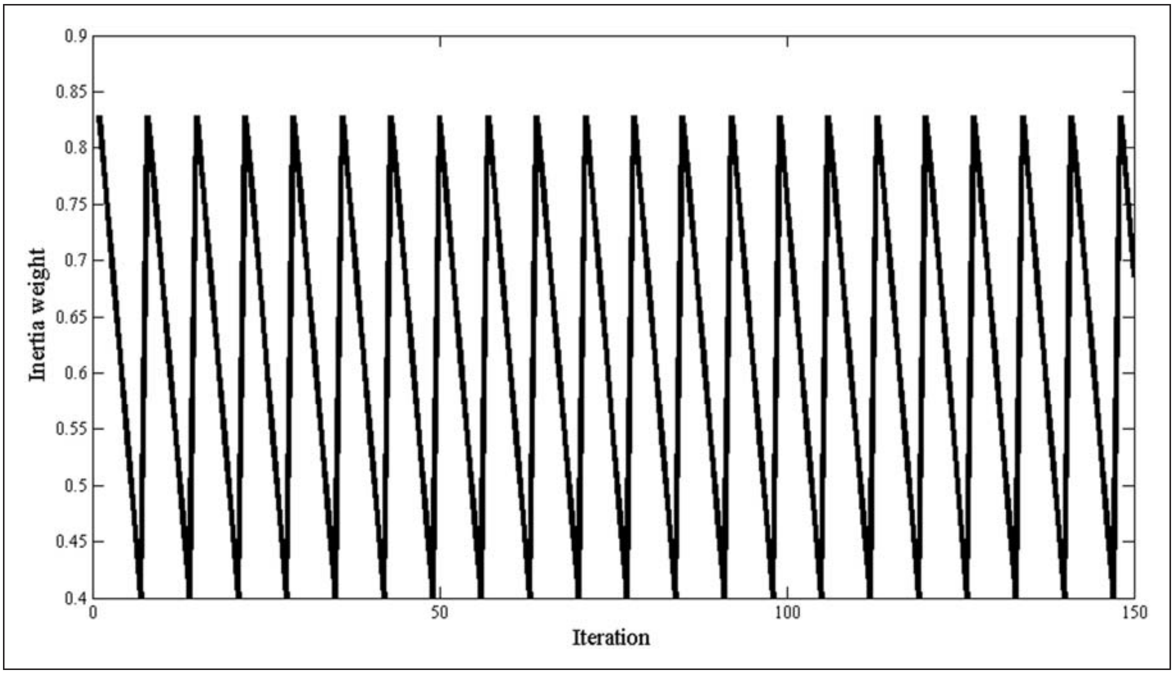

Figure 1. Behavior of the inertia weight in the periodic multi-objective CDPSO algorithm.

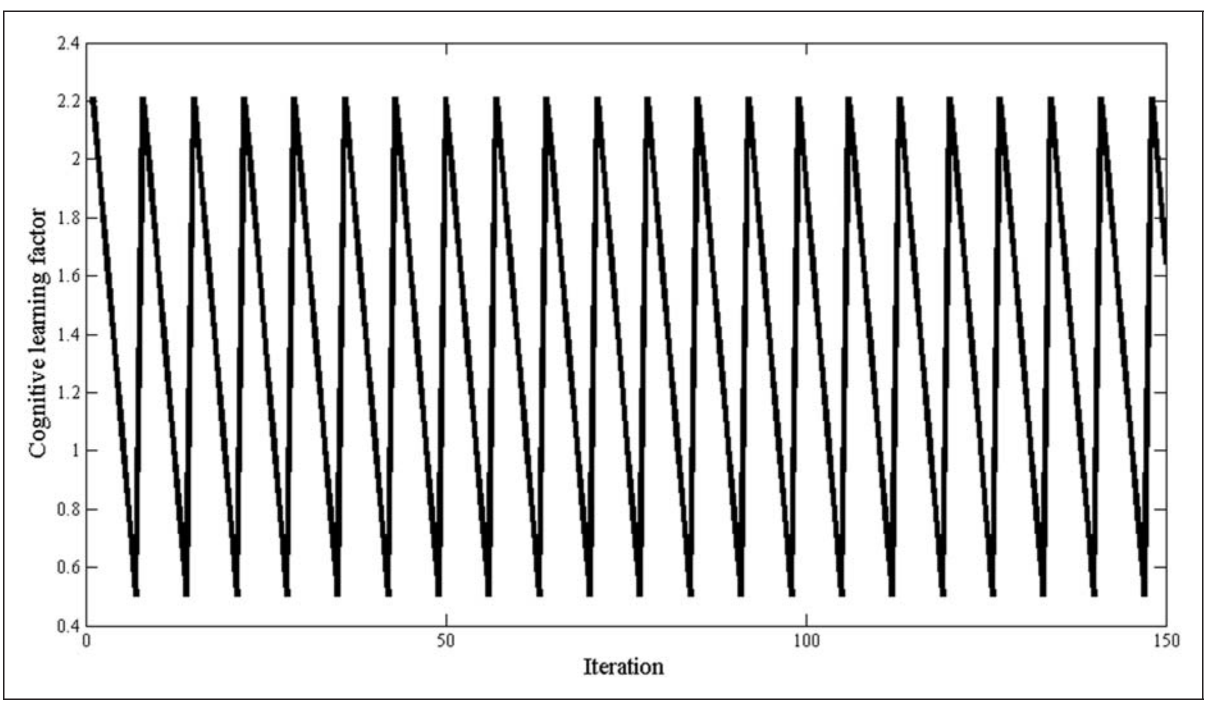

Figure 2. Behavior of the cognitive learning factor in the periodic multi-objective CDPSO algorithm.

2011, 2012). In this approach, all non-dominated solutions have an elimination radius equal to $\varepsilon_{elimination}$, and if the Euclidean distance between two particles in the archive is less than $\varepsilon_{elimination}$, then one of them will be neglected. Moreover, the following equation was established to compute the value of $\varepsilon_{elimination}$:

$$\varepsilon_{elimination} = \frac{t}{\zeta \times \text{maximum iteration}} \tag{13}$$

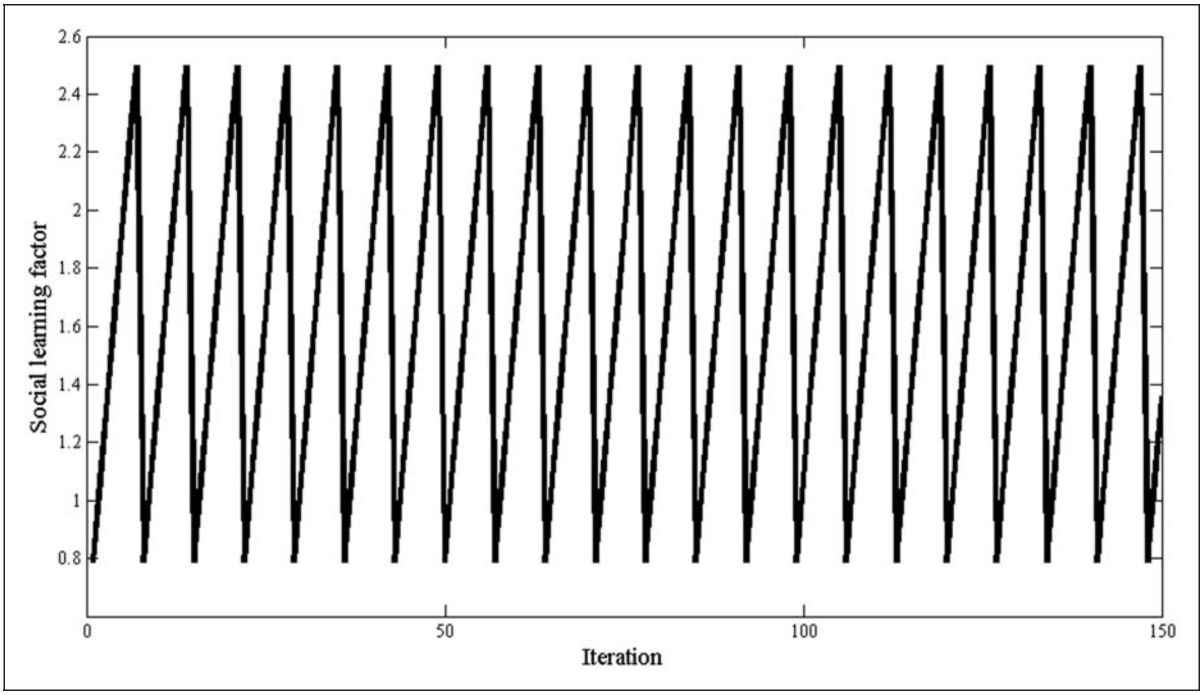

Figure 3. Behavior of the social learning factor in the periodic multi-objective CDPSO algorithm.

where ζ is a positive constant, t and maximum iteration are the current and maximum number of iteration, respectively.

5. Fuzzy controller

Zadeh (1968) originally proposed fuzzy logic and fuzzy set theory in the 1970s, and these have also received considerable attention among researchers as a new topic. Fuzzy systems are knowledge-based or rule-based systems formed via human knowledge and heuristics. They have been applied to a wide range of research fields such as control, communication, medicine, management, business, psychology, etc. The most significant applications and studies about fuzzy systems have concentrated on control areas (such as those in Kim and Bien, 2000; Koo, 2001; Samanta and Al-Araimi, 2001; Bezine et al., 2002; Kuo and Lin, 2002; Xu and Xu, 2003; Sakhare et al., 2004; Li et al., 2005; Labiod et al., 2005; Hay et al., 2005; Ghasem, 2006; Lygouras et al., 2007; Castillo et al., 2008; Saad and Zellouma, 2009; Jamaludin et al., 2009; Tzafestas et al., 2010; Liu et al., 2010; Wang et al., 2011; Ishaque et al., 2011; Cazarez-Castro et al., 2012; Nagi et al., 2013). In particular, Yubazaki et al. (1996) proposed a single input rule modules (SIRMs) dynamically connected fuzzy inference model for plural input fuzzy control. After that, they successfully employed the proposed technique to control the inverted pendulum (Yi and Yubazaki, 2000; Yi et al., 2001a), ball and beam (Yi et al., 2001b), series-type double inverted pendulum (Yi et al., 2001c), parallel-type double inverted pendulum (Yi et al., 2002), etc. This controller takes the system states as input data, and the driving force as the output item. In essence, for each input, a SIRM and a dynamic importance degree are made. The dynamic importance degree contains a base value and a dynamic value. The base value identifies the role of the related input through a control process, and the dynamic value changes with control situations to regulate the dynamic importance degree. Furthermore, to tune the control parameters automatically, the random optimization search method was used. In the following subsections, the single input rule modules and the dynamic importance degree are briefly described.

5.1. Single input rule modules (SIRMs)

Consider a system with n inputs and one output. Because there are n input items, n SIRMs would be created as the following:

$$\text{SIRM}_i: \quad \{R_i^j : \text{if } x_i = A_i^j \text{ then } u_i = C_i^j\}_{j=1}^{m_i} \tag{14}$$

Each SIRM relates separately to one of the n inputs. Hence, SIRM_i means the SIRM referred to the ith input item, and R_i^j is the jth rule in the SIRM_i. x_i shows the ith input item and is the variable in the antecedent part. u_i is the variable in the consequent part of the SIRM_i. A_i^j and C_i^j are the membership functions of the x_i and u_i in the jth rule of the SIRM$_i$, respectively. Further, $i = 1, 2, \ldots, n$ is the index number of the SIRMs, and $j = 1, 2, \ldots, m$ is the index number of the rules in the SIRM$_i$. Here, the rules and membership functions of the SIRMs for the inverted pendulum and ball-beam system are according to Yi and Yubazaki (2000) and Yi et al. (2001b), respectively.

5.2. Dynamic importance degree

To implement the different role of each input in the controller effort, a dynamic importance degree p_i^D independently defines for each input item $x_i (i = 1, 2, \ldots, n)$ as follows:

$$p_i^D = p_i^0 + Q_i \Delta p_i^0 \tag{15}$$

where p_i^0 is the base value, Δp_i^0 is the dynamic variable, and Q_i is the breadth for input ith $(i = 1, 2, \ldots, n)$. The dynamic variable Δp_i^0 would be calculated via fuzzy rules. The base value p_i^0 and the breadth Q_i are control parameters and could be obtained by trial and error. In this paper, the dynamic variable Δp_i^0 of the inputs $x_i (i = 1, 2, \ldots, n)$ for the inverted pendulum and ball-beam systems are completely similar to Yi and Yubazaki (2000) and Yi et al. (2001b), respectively.

When the dynamic importance degrees and the variables in the consequent part of the SIRM$_i$ are determined, the driving force F could be obtained by equation (16).

$$F = OSF \times u = OSF \times \sum_{i=1}^{n} p_i^D u_i^0 \tag{16}$$

where OSF is the output scaling factor, u is the output item of the SIRMs dynamically connected fuzzy inference model, and p_i^D is the dynamic importance degree. u_i^0 is the fuzzy inference result of the consequent variable u_i. u_i^0 can be calculated via the min-max-gravity or product-sum-gravity or simplified inference methods.

In this paper, the base value p_i^0 and the breadth Q_i would be determined using the periodic multi-objective CDPSO algorithm.

6. Pareto design of sirms dynamically connected fuzzy controller

In this section, the Pareto design of the SIRMs dynamically connected fuzzy controllers using the periodic multi-objective algorithm and the multi-objective genetic algorithm (http://www.mathworks.com) is performed for inverted pendulum and ball-beam systems. The errors of the system states are considered as the objective functions. These objective functions have to be minimized simultaneously. The parameters of the multi-objective CDPSO algorithm are adjusted based on the experimental results in Mahmoodabadi et al. (2011, 2012) as the following. $W_1 = 0.9$, $W_2 = 0.4$, $C_{1i} = 2.5$, $C_{1f} = 0.5$, $C_{2i} = 0.5$, $C_{2f} = 2.5$, the population size 100, the maximum iteration 1000, and the number of iterations in a period $T = 7$ are used. Furthermore, the positive constant to prune the archive is $\zeta = 300$ and the neighborhood radius for the leader selection is $R_{neighborhood} = 0.01$. The parameter configurations of the multi-objective genetic algorithm (MOGA) are: the population size is 100, the maximum generation is 1000, the pruning of the archive is based on distance crowding, the crossover method is scattered and the mutation approach is constraint dependent.

6.1. Inverted pendulum system

The inverted pendulum system is a complicated nonlinear and unstable system of high order. The mechanical structure of the inverted pendulum is illustrated in Figure 4. The system observable state vector is $x = [x, \dot{x}, \theta, \dot{\theta}] = [x_1, x_2, x_3, x_4]$ and includes the cart position, the cart velocity, the pendulum angle and the pendulum angular velocity, respectively. The mathematical model of the system is given by:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{\frac{4}{3}\left[F + m_p l_p x_4^2 \sin(x_3)\right] - m_p g \sin(x_3)\cos(x_3)}{\frac{4}{3}\left(m_c + m_p\right) - m_p \cos^2(x_3)} \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \frac{\left(m_c + m_p\right) g \sin(x_3) - \left[F + m_p l_p x_4^2 \sin(x_3)\right]\cos(x_3)}{\left[\frac{4}{3}\left(m_c + m_p\right) - m_p \cos^2(x_3)\right] l_p}
\end{aligned} \tag{17}$$

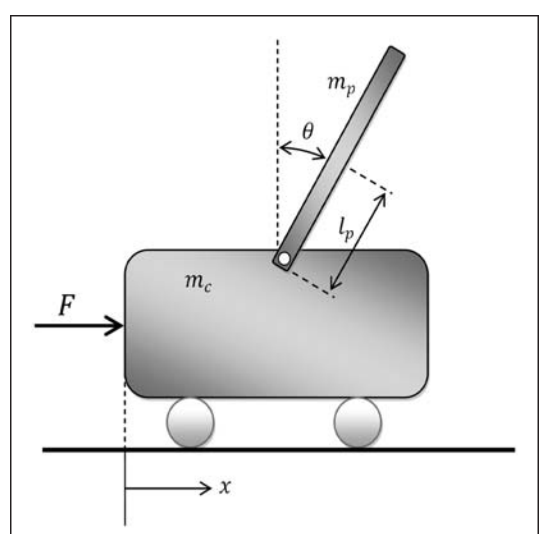

Figure 4. Mechanical structure of the inverted pendulum system.

where m_c is the mass of the cart, m_p is the mass of the pendulum, g is the gravity acceleration, and F is the driving force. l_p is the length from the center of the pendulum to the pivot and is equal to the half length of the pendulum. For simulation, the following specifications are used:

$$m_c = 1\,kg,\ m_p = 0.1\,kg,\ l_p = 0.5\,m,\ \text{and}\ g = 9.8\,\tfrac{m}{s^2}.$$

The initial and desired conditions are $x = [x_1, x_2, x_3, x_4] = [2, 0, 0, 0]$ and $x = [x_1, x_2, x_3, x_4] = [0, 0, 0, 0]$, respectively. The vector $[p_1^0, p_2^0, p_3^0, p_4^0,\ Q_1,\ Q_2, Q_3, Q_4, OSF]$ is the vector of design variables of SIRMs dynamically connected fuzzy controllers. $p_i^0 (i = 1, 2, 3, 4)$, Q_i $(i = 1, 2, 3, 4)$, and OSF are the base values, the breadths, and the output scaling factor, respectively. The normalized angle error of the pendulum and the normalized distance error of the cart are considered as the objective functions that have to be minimized, simultaneously. The feasibility and efficiency of multi-objective CDPSO is assessed in comparison with MOGA and the suggested point by Yi and Yubazaki (2000) (Figure 5). This figure shows that the obtained solutions by periodic CDPSO have better convergence and more uniform distribution in comparison with MOGA results. Furthermore, it can be clearly seen that the suggested point by Yi and Yubazaki (2000) is dominated by some points of the Pareto front. It demonstrates the effectiveness of this work to obtain the optimum points (compare objective functions of points B and D in Table 1).

In Figure 5, points A and C show the best normalized distance error of the cart and the normalized angle error of the pendulum, respectively. The green color points are the equitable solutions of the proposed

Table 1. The objective functions and design variables of the optimum points shown in Figure 5.

Optimum design points	A	B	C	D
p_1^0	5.807	7.202	12.358	2.00
p_2^0	4.749	3.516	1.975	1.50
p_3^0	0.4943	0.1899	0.0870	0.15
p_4^0	0.6410	0.5877	0.3032	0.15
Q_1	6.891	−2.671	−12.008	2.50
Q_2	−9.992	−1.903	2.200	1.00
Q_3	0.9680	0.9285	0.3124	0.20
Q_4	0.3756	0.5241	0.3652	0.20
OSF	2.341	2.873	7.499	11
The normalized distance error of the cart	0.3781	0.4940	0.9881	0.5031
The normalized angle error of the pendulum	0.9845	0.7081	0.2702	0.7380
Maximum driving force (N)	2.853	2.678	2.496	3.208

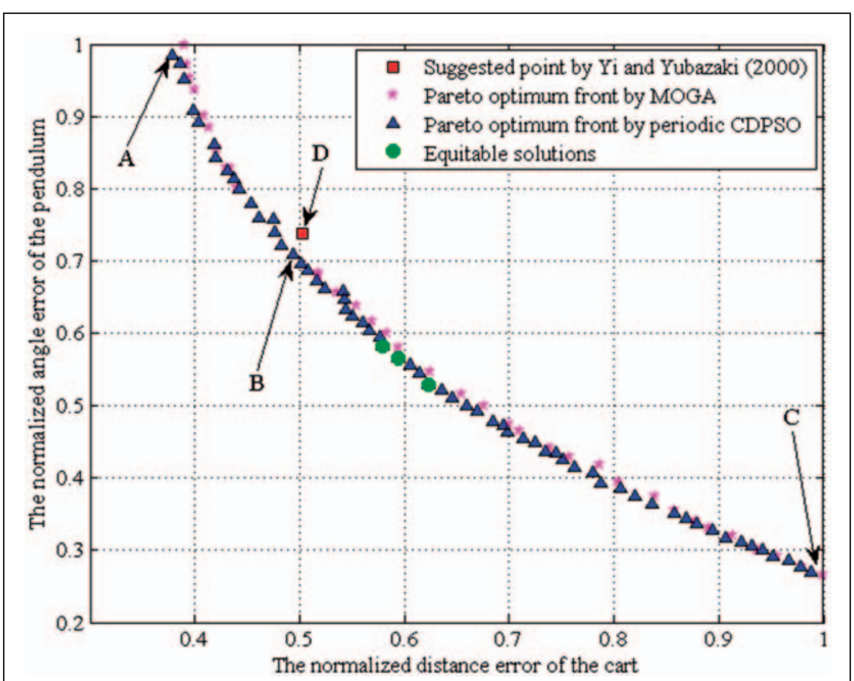

Figure 5. Optimum design points of the fuzzy controller for the inverted pendulum system.

Pareto front obtained by the Lorenz dominance methodology. The interesting result is that the proposed algorithm has several points dominated to point D (the suggested point by Yi and Yubazaki, 2000) with less maximum driving force, such as point B (Figure 5 and Table 1). Design variables, objective functions, and maximum driving forces related to the optimum design points A, B, C, and D are presented in Table 1. Moreover, the time responses of the pendulum and cart related to these points are shown in Figures 6 and 7. It is clear from these figures that point A has the best distance control of the cart and the worse angular control of the pendulum, while point C has the best angular control of the pendulum and the worst distance control of the cart. Furthermore, point B has the better distance control of the cart and the angular control of the pendulum in comparison with point D. Figure 8 shows the driving forces of the optimum design points A, B, C, and D.

6.2. Ball-beam system

Here, we consider the ball-beam system depicted in Figure 9. The state vector is the system observable state vector $x = [x, \dot{x}, \theta, \dot{\theta}] = [x_1, x_2, x_3, x_4]$, including, respectively, the ball position, the ball velocity, the beam angle, and the beam angular velocity. In addition, its dynamic is described as follows:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= B\,[x_1 x_4^2 - g \sin x_3] \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= u \end{aligned} \tag{18}$$

where M is the ball mass, g is the gravity acceleration, J_1 is the ball inertia moment, and J_2 is the beam inertia moment. The manipulated variable u is related with torque τ by:

$$\tau = 2Mx_1x_2x_4 + gMx_1 \cos x_3 + (J_1 + J_2 + Mx_1^2)u \tag{19}$$

The system parameters used for simulation are $M = 0.05$ kg, $J_1 = 2 \times 10^{-6}$ kgm^2, $J_2 = 0.02$ kgm^2, $g = 9.81\ \frac{m}{s^2}$, and $B = 0.7143$. Also, the scaling factor is regarded as $OSF = 1$.

The vector $[p_1^0, p_2^0, p_3^0, p_4^0,\ Q_1, Q_2, Q_3, Q_4]$ is the vector of selective parameters (design variables) of SIRMs dynamically connected fuzzy controllers.

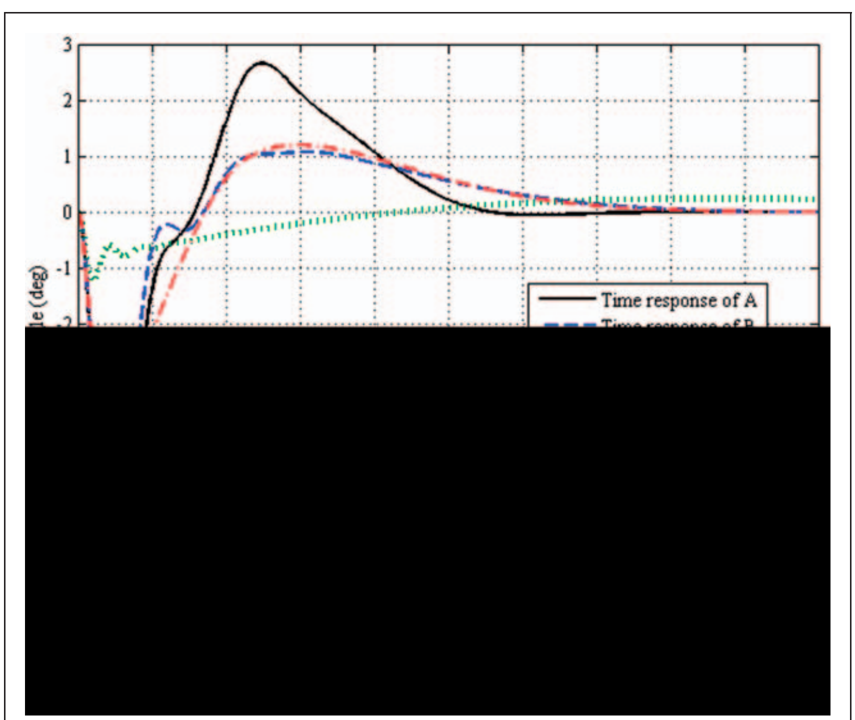

Figure 6. Time responses of the pendulum for optimum design points shown in Figure 5.

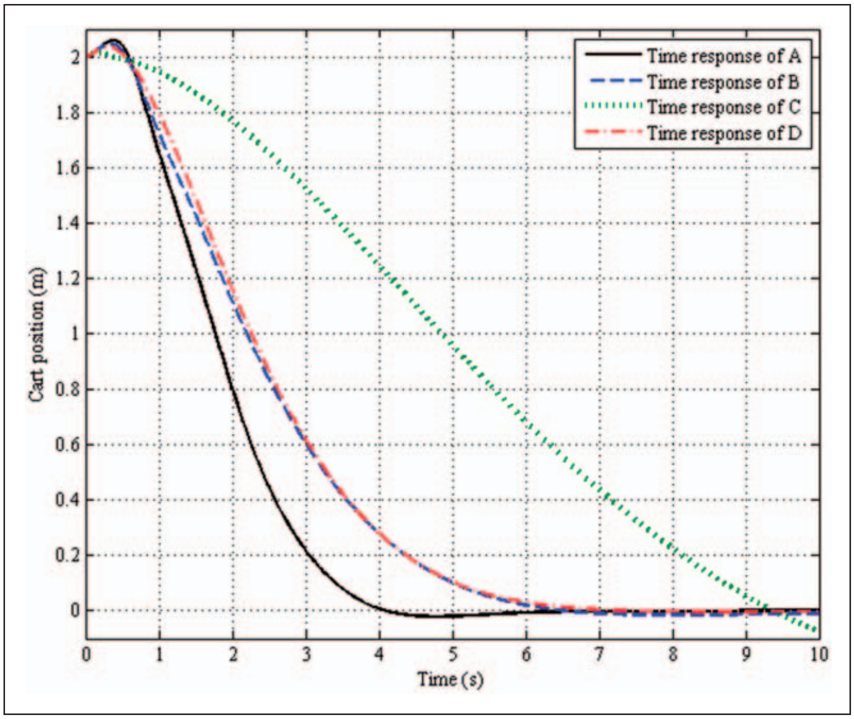

Figure 7. Time responses of the cart for the optimum design points shown in Figure 5.

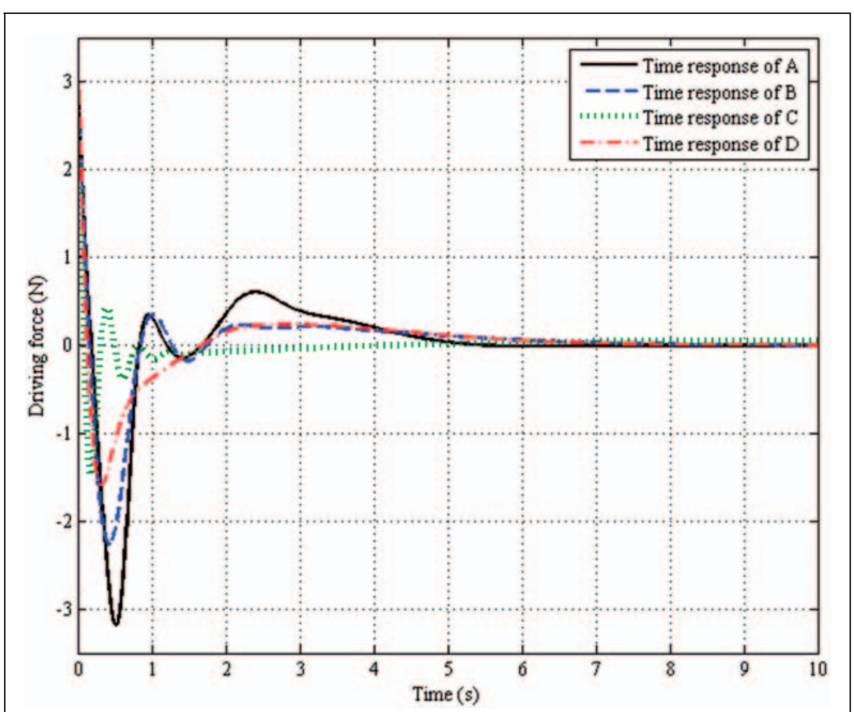

Figure 8. Driving forces of the optimum design points shown in Figure 5.

$p_i^0(i = 1, 2, 3, 4)$ and $Q_i(i = 1, 2, 3, 4)$ are the base values and the breadths, respectively. The angle error of the beam and the distance error of the ball are functions of this vector. This means that by selecting the design variables, we are able to change the angle and distance errors. Obviously, this is an optimization problem with two object functions (angle error of the beam and distance error of the ball) and eight design variables $[p_1^0, p_2^0, p_3^0, p_4^0, \quad Q_1, Q_2, Q_3, Q_4]$. The initial and desired values are $x = [x_1, x_2, x_3, x_4] = \left[0.5, 0, \frac{\pi}{6}, 0\right]$ and $x = [x_1, x_2, x_3, x_4] = [0, 0, 0, 0]$, respectively. The Pareto front obtained from the proposed method is compared with the Pareto front of the multi-objective genetic algorithm in Figure 10. It can be clearly observed from this figure that the proposed Pareto front achieves better and more diverse objective functions than MOGA for the present case study, which demonstrates the effectiveness of this work. Moreover, the suggested point D by Yi et al. (2001b) is drastically dominated by all points of the Pareto sets shown in Figure 10.

In Figure 10, points C and A stand for the best angle error of the beam and distance error of the ball, respectively. It can be seen from this figure that all the optimum design points in the Pareto front are non-dominated and could be chosen by a designer as optimum SIRMs dynamically connected fuzzy controllers.

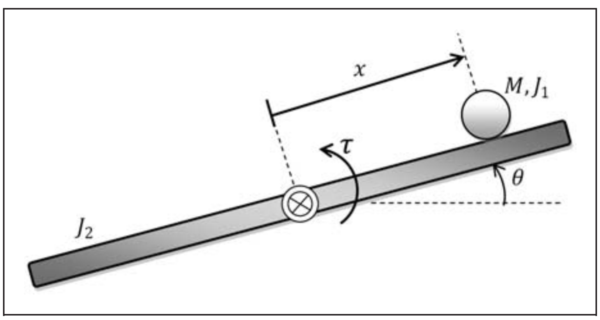

Figure 9. Mechanical structure of the ball-beam system.

Table 2. The values of objective functions and their associated design variables for the optimum points shown in Figure 10.

Optimum design points	A	B	C	D
p_1^0	6.6219	5.8302	6.9610	4.6552
p_2^0	7.5631	8.1254	8.2052	6.9418
p_3^0	25.9811	25.0777	23.9687	23.3023
p_4^0	10.7181	13.2139	11.2948	15.3286
Q_1	6.9709	4.9164	3.0613	6.9432
Q_2	5.0025	7.1920	9.9899	7.0228
Q_3	3.5753	3.0969	1.0830	5.1883
Q_4	2.0002	2.0000	6.5392	6.4724
The distance error of the ball	0.2130	0.2328	0.2723	0.3220
The angle error of the beam	0.3529	0.2653	0.2532	0.5037
Maximum torque	0.2946	0.2810	0.2117	0.2918

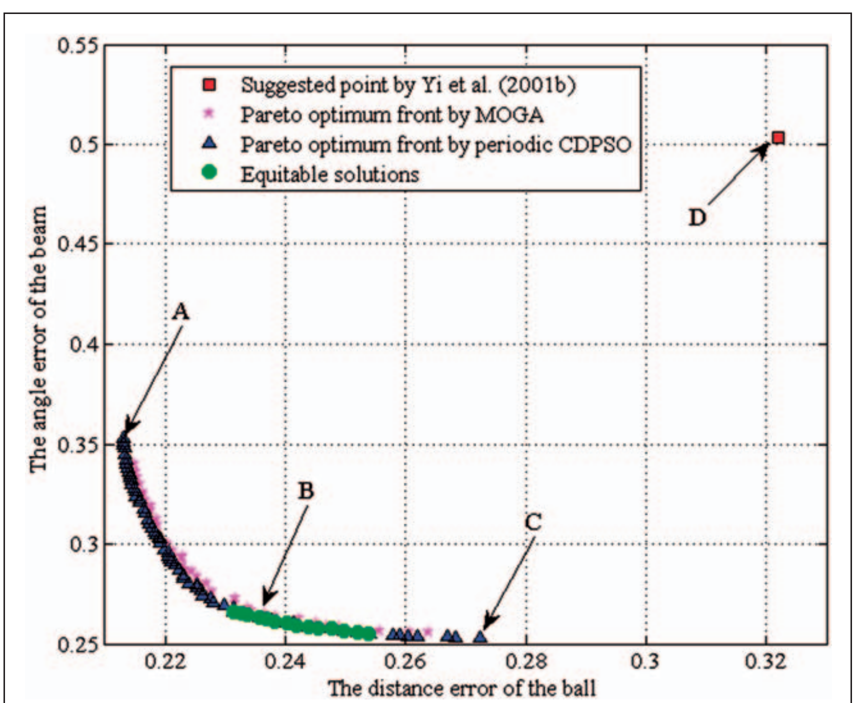

Figure 10. Optimum design points of the fuzzy controller for the ball-beam system.

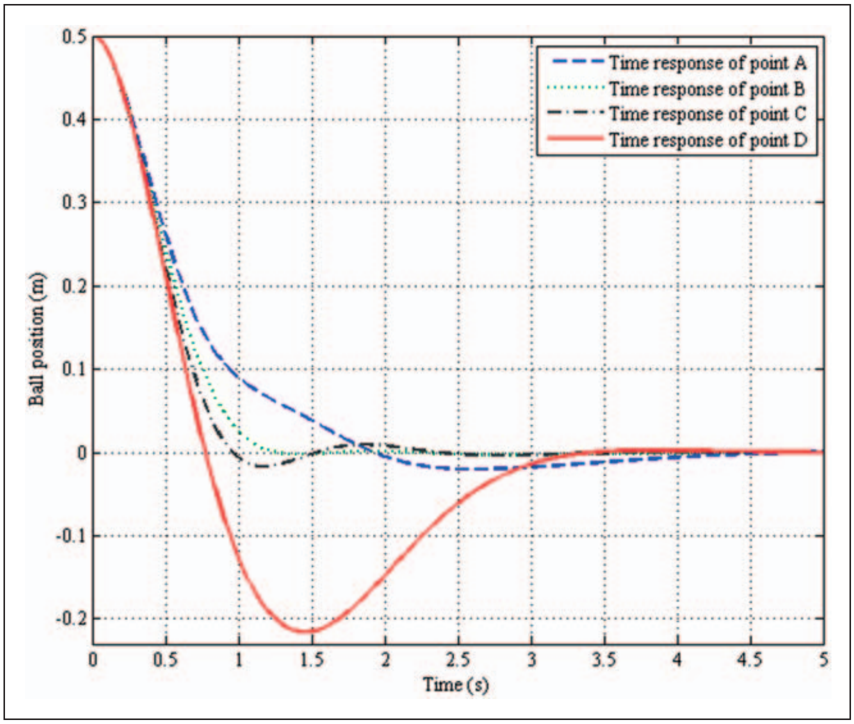

Figure 11. Time responses of the ball for optimum design points shown in Figure 10.

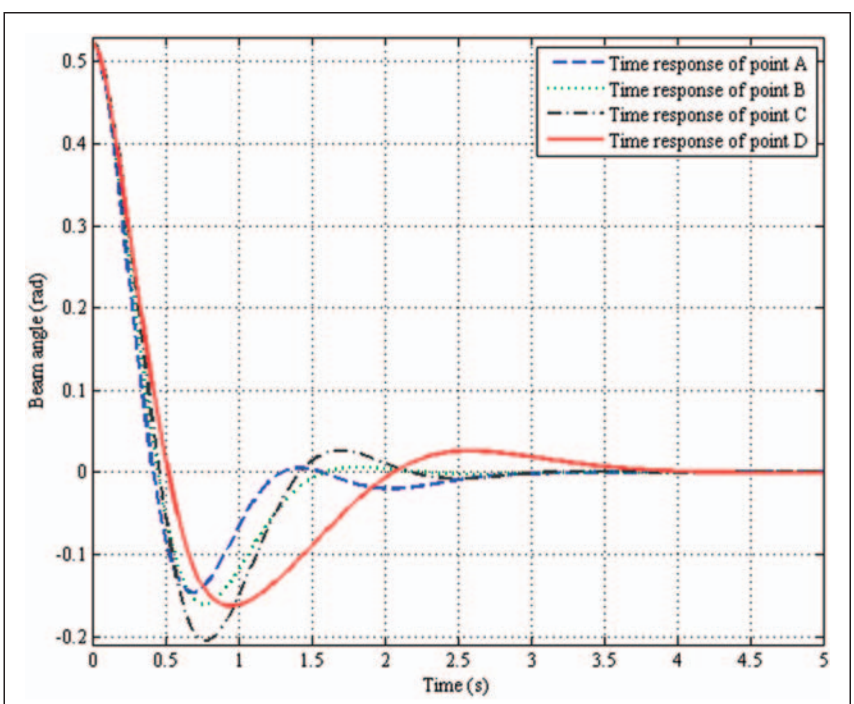

Figure 12. Time responses of the beam for optimum design points shown in Figure 10.

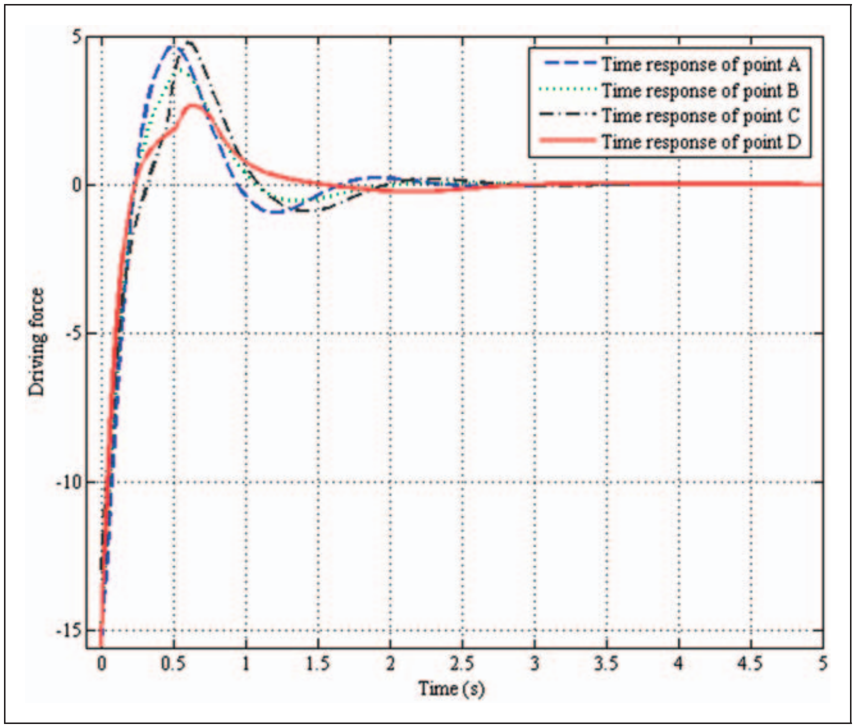

Figure 13. Driving forces of the optimum design points shown in Figure 10.

The corresponding design variables (vector of SIRMs dynamically connected fuzzy controllers) of the Pareto front shown in Figure 10 are the pre-eminent possible design points. This figure illustrates that if any other set of design variables is chosen, the corresponding values of the pair of those objective functions will locate a point inferior to the Pareto front. Such a design fact could not have been discovered without the use of the multi-objective Pareto optimization process. Furthermore, in this figure, the green color points are the equitable solutions of the proposed Pareto front obtained by the Lorenz dominance approach. The most interesting result is that the suggested point D by Yi et al. (2001b) is drastically dominated by all Pareto optimal points (Table 2). Design variables, objective functions, and maximum driving forces related to the optimum design points A, B, C, and D are stated in Table 2. The time responses related to these points are shown in Figures 11, 12, and 13.

7. Conclusion

In this study, a multi-objective particle swarm optimization algorithm combined with convergence and divergence operators was successfully used to optimally design SIRMs dynamically connected fuzzy controllers for the inverted pendulum and ball-beam systems. The conflicting objective functions for the inverted pendulum system were the normalized angle error of the pendulum and the normalized distance error of the cart; and for the ball-beam system, they were the angle error of the beam and the distance error of the ball. The reported results demonstrated that the proposed methodology can effectively control the nonlinear systems.

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