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Optimal admission control for two station tandem queues with loss

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ABSTRACT

We consider a two-station tandem queue with a buffer size of one at the first station and a finite buffer size at the second station. Silva et al. (2013) gave a criterion determining the optimal admission control policy for this model. In this paper, we improve the results of Silva et al. (2013) and also solve the problem conjectured by Silva et al. (2013).

1. Introduction

We consider a tandem queueing network with two stations (station 1 and station 2) studied by Silva et al. [6]. There is one server at each station, and customers arrive to station 1 according to a Poisson process with rate λ. The service times of the customers at station i are independently and exponentially distributed with rate $\mu_i, i = 1, 2$. The size of the buffer (which includes the customer in service and customers waiting) at station 1 is one, i.e., customers are not allowed to wait for service at station 1, while the buffer size at station 2 is B, where $1 \leq B < \infty$. A gatekeeper who has complete knowledge of the number of customers at both stations decides to admit or reject each arrival. If an arrival is not admitted, a cost c_1 is incurred. If station 1 is full at the time of an arrival, then the gatekeeper has to reject the incoming customer. If an arrival is accepted, an arriving customer receives service at station 1. Once a customer completes service at station 1, the customer proceeds to station 2. At station 2, if the customer finds it empty the customer receives service at station 2 and eventually leaves the system, but if station 2 is full, the customer is lost and a cost c_2 is incurred. The objective for the gatekeeper is to make optimal admission decisions in order to minimize the long-run average cost.

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Zhang and Ayhan [8] considered a tandem queueing network with two stations. The first station has a finite buffer size and the second station has a unitary buffer size. They studied the optimal admission control policy for minimizing the long-run average cost. Many researchers have studied control problems for tandem queues with loss, see, for example, Chang and Chen [1], Ku and Jordan [2–4], Sheu and Ziedins [5] and Spicer and Ziedins [7].

This paper is inspired by the work of Silva et al. [6] who studied the optimal policy that minimizes the long-run average cost for the same model as the one considered here. Silva et al. [6] showed that there are only two policies that could be optimal. The first policy is the prudent policy, denoted by π_P. Under a prudent policy, the gatekeeper admits an arrival whenever station 1 is empty and the number of customers at station 2 is less than B. The second policy is the greedy policy, denoted by π_G. Under a greedy policy the gatekeeper admits an arrival whenever possible (i.e., whenever station 1 is empty) and rejects otherwise. Silva et al. [6] gave a criterion that determines which of the two policies is optimal. The optimality condition is of the threshold type (on the cost c_2) and the threshold is expressed in terms of the stationary distributions under the prudent and the greedy policies. They provided closed-form expressions for the threshold when $B \leq 10$, but mentioned that it is difficult to obtain a closed-form expression for the threshold when $B > 10$ because there is no closed-form expression for the stationary distributions for general values of B. They also made a conjecture about the monotonicity properties of the stationary distributions.

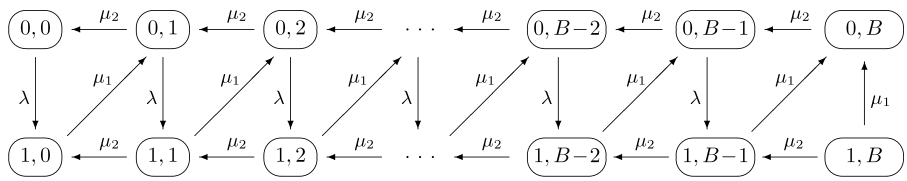

Fig. 1. The transition diagram of the Markov process $\{(N_1(t), N_2(t)) : t \ge 0\}$ under the prudent policy.

In this paper, we obtain closed-form expressions for the stationary distributions of the Markov processes describing the systems under the prudent and the greedy policies for any values of B. We explicitly determine the criterion for the optimal policy by providing a closed-form expression of the threshold (on the cost c_2) for any values of B. The threshold is expressed in terms of the parameters of the model, i.e., the service rates at both stations, the arrival rate, the cost c_1, and the buffer size. We also solve the problem about the monotonicity properties of the stationary distributions, conjectured by Silva et al. [6].

2. Stationary distributions under prudent and greedy policies

In this section we not only obtain the stationary distributions under the prudent and the greedy policies, but also solve the conjecture of Silva et al. [6], by introducing the scaled stationary probabilities and investigating the properties of this scaled probabilities.

2.1. Prudent policy

Let $N_i(t)$, $i = 1, 2$, denote the number of customers (including those waiting and in service) at station i at time t, in the system operating under the prudent policy. Then $\{(N_1(t), N_2(t)) : t \ge 0\}$ is a continuous time Markov process with state space $\mathcal{S} = \{(i, j) : i = 0, 1, j = 0, 1, \ldots, B\}$. Fig. 1 depicts the state transition diagram of the Markov process $\{(N_1(t), N_2(t)) : t \ge 0\}$.

Let $p^{(B)}_{(i,j)}$, $(i, j) \in \mathcal{S}$ be the stationary distribution of the Markov process $\{(N_1(t), N_2(t)) : t \ge 0\}$, i.e.,

$$p^{(B)}_{(i,j)} = \lim_{t\to\infty} \mathbb{P}((N_1(t), N_2(t)) = (i, j)).$$

Define

$$x^{(B)}_n \equiv \frac{p^{(B)}_{(0,B-n)}}{p^{(B)}_{(0,B)}}, \qquad y^{(B)}_n \equiv \frac{p^{(B)}_{(1,B-n)}}{p^{(B)}_{(0,B)}},$$

for $n = 0, 1, \ldots, B$.

Lemma 1. *We have the following recurrence formula:*

$$\begin{bmatrix} x^{(B)}_0 \\ y^{(B)}_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{1}$$

$$\begin{bmatrix} x^{(B)}_{n+1} \\ y^{(B)}_{n+1} \end{bmatrix} = \begin{bmatrix} \dfrac{(\mu_1+\mu_2)\mu_2}{\lambda\mu_1} & \dfrac{\mu_2^2}{\lambda\mu_1} \\ \dfrac{\mu_2}{\mu_1} & \dfrac{\mu_2}{\mu_1} \end{bmatrix} \begin{bmatrix} x^{(B)}_n \\ y^{(B)}_n \end{bmatrix}, \quad n = 0, 1, \ldots, B-2, \tag{2}$$

$$\begin{bmatrix} x^{(B)}_B \\ y^{(B)}_B \end{bmatrix} = \begin{bmatrix} \dfrac{\mu_2}{\lambda} & 0 \\ \dfrac{\mu_2}{\mu_1} & \dfrac{\mu_2}{\mu_1} \end{bmatrix} \begin{bmatrix} x^{(B)}_{B-1} \\ y^{(B)}_{B-1} \end{bmatrix}. \tag{3}$$

Proof. Note that (1) is trivial. We can prove (2) and (3) by balancing the transition rates into and out of a particular set of states. Let $q^{(B)}_{(\mathbf{s},\mathbf{s}')}$, $\mathbf{s}, \mathbf{s}' \in \mathcal{S}$ be the transition rates of the Markov process $\{(N_1(t), N_2(t)) : t \ge 0\}$ from state $\mathbf{s}$ to $\mathbf{s}'$. We partition the state space into two disjoint subsets E and E^c. Then we equate transition rates from one subset to the other:

$$\sum_{\mathbf{s}\in E}\sum_{\mathbf{s}'\in E^c} p^{(B)}_{\mathbf{s}} q^{(B)}_{(\mathbf{s},\mathbf{s}')} = \sum_{\mathbf{s}\in E^c}\sum_{\mathbf{s}'\in E} p^{(B)}_{\mathbf{s}} q^{(B)}_{(\mathbf{s},\mathbf{s}')}. \tag{4}$$

By choosing $E = \{(0, 0), (0, 1), \ldots, (0, B-(n+1)), (1, 0), (1, 1), \ldots, (1, B-(n+1))\}$, $n = 0, 1, \ldots, B-2$, we have from (4) that $\mu_1 p^{(B)}_{(1,B-n-1)} = \mu_2\big(p^{(B)}_{(0,B-n)} + p^{(B)}_{(1,B-n)}\big)$, or equivalently,

$$y^{(B)}_{n+1} = \frac{\mu_2}{\mu_1}(x^{(B)}_n + y^{(B)}_n). \tag{5}$$

Similarly, choosing $E = \{(0, 0), (0, 1), \ldots, (0, B-(n+1)), (1, 0), (1, 1), \ldots, (1, B-(n+2))\}$, $n = 0, 1, \ldots, B-2$, yields $\lambda p^{(B)}_{(0,B-n-1)} = \mu_2\big(p^{(B)}_{(0,B-n)} + p^{(B)}_{(1,B-n-1)}\big)$, or equivalently,

$$\lambda x^{(B)}_{n+1} = \mu_2 x^{(B)}_n + \mu_2 y^{(B)}_{n+1}.$$

Plugging (5) into the above, we get

$$x^{(B)}_{n+1} = \frac{(\mu_1+\mu_2)\mu_2}{\lambda\mu_1} x^{(B)}_n + \frac{\mu_2^2}{\lambda\mu_1} y^{(B)}_n.$$

From this and (5), we have (2).

Now we prove (3). By choosing $E = \{(0, 0)\}$, we have from (4) that $\lambda p^{(B)}_{(0,0)} = \mu_2 p^{(B)}_{(0,1)}$, or equivalently,

$$x^{(B)}_B = \frac{\mu_2}{\lambda} x^{(B)}_{B-1}. \tag{6}$$

Similarly, choosing $E = \{(1, 0)\}$ yields $\mu_1 p^{(B)}_{(1,0)} = \lambda p^{(B)}_{(0,0)} + \mu_2 p^{(B)}_{(1,1)}$, or equivalently,

$$\mu_1 y^{(B)}_B = \lambda x^{(B)}_B + \mu_2 y^{(B)}_{B-1}.$$

Plugging (6) into the above, we have

$$y^{(B)}_B = \frac{\mu_2}{\mu_1} x^{(B)}_{B-1} + \frac{\mu_2}{\mu_1} y^{(B)}_{B-1}.$$

This and (6) give (3). □

Lemma 2. *Let $S^{(B)} = \sum_{n=0}^{B}(x^{(B)}_n + y^{(B)}_n)$. Then*

(i) *$S^{(B)}$ is strictly increasing in B.*
(ii) *$\lim_{B\to\infty} S^{(B)} < \infty$ if and only if $\chi_1 < 1$, where $\chi_1 = \frac{\mu_2}{2\lambda\mu_1}\big(\lambda + \mu_1 + \mu_2 + \sqrt{(\lambda-\mu_1)^2 + 2(\lambda+\mu_1)\mu_2 + \mu_2^2}\big)$.*
In that case,

$$\lim_{B\to\infty} S^{(B)} = \frac{\lambda\mu_1}{\lambda(\mu_1-\mu_2) - \mu_1\mu_2}.$$

Proof. Note that for $n = 0, 1, \ldots, B-1$,

$$x^{(B+1)}_n = x^{(B)}_n, \qquad y^{(B+1)}_n = y^{(B)}_n.$$

From this and Lemma 1, we have

$$\begin{aligned} S^{(B+1)} - S^{(B)} &= x_B^{(B+1)} + y_B^{(B+1)} + x_{B+1}^{(B+1)} + y_{B+1}^{(B+1)} - (x_B^{(B)} + y_B^{(B)}) \\ &> (x_B^{(B+1)} + y_B^{(B+1)}) - (x_B^{(B)} + y_B^{(B)}) \\ &= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{(\mu_1+\mu_2)\mu_2}{\lambda\mu_1} & \frac{\mu_2^2}{\lambda\mu_1} \\ \frac{\mu_2}{\mu_1} & \frac{\mu_2}{\mu_1} \end{bmatrix} \begin{bmatrix} x_{B-1}^{(B)} \\ y_{B-1}^{(B)} \end{bmatrix} \\ &\quad - \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{\mu_2}{\lambda} & 0 \\ \frac{\mu_2}{\mu_1} & \frac{\mu_2}{\mu_1} \end{bmatrix} \begin{bmatrix} x_{B-1}^{(B)} \\ y_{B-1}^{(B)} \end{bmatrix} \\ &= \frac{\mu_2^2}{\lambda\mu_1}(x_{B-1}^{(B)} + y_{B-1}^{(B)}) > 0. \end{aligned}$$

This proves assertion (i).

Now we prove assertion (ii). By Lemma 1, we have

$$\lim_{B\to\infty} S^{(B)} = \begin{bmatrix} 1 & 1 \end{bmatrix} \sum_{n=0}^{\infty} \begin{bmatrix} \frac{(\mu_1+\mu_2)\mu_2}{\lambda\mu_1} & \frac{\mu_2^2}{\lambda\mu_1} \\ \frac{\mu_2}{\mu_1} & \frac{\mu_2}{\mu_1} \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{7}$$

We know that for a matrix A of finite order the series $\sum_{n=0}^{\infty} A^n$ converges if and only if the spectral radius of A is strictly less than 1. In that case, $\sum_{n=0}^{\infty} A^n = (I-A)^{-1}$. For simplicity, we denote

$$\mathbf{e} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad M = \begin{bmatrix} \frac{(\mu_1+\mu_2)\mu_2}{\lambda\mu_1} & \frac{\mu_2^2}{\lambda\mu_1} \\ \frac{\mu_2}{\mu_1} & \frac{\mu_2}{\mu_1} \end{bmatrix},$$

then (7) can be written as

$$\lim_{B\to\infty} S^{(B)} = \mathbf{e}^\top \sum_{n=0}^{\infty} M^n \mathbf{e}_1,$$

where $\top$ denotes the transpose of a vector. The matrix M has two eigenvalues

$$\chi_1 = \frac{\mu_2}{2\lambda\mu_1}\left(\lambda + \mu_1 + \mu_2 + \sqrt{(\lambda-\mu_1)^2 + 2(\lambda+\mu_1)\mu_2 + \mu_2^2}\right), \tag{8}$$

$$\chi_2 = \frac{\mu_2}{2\lambda\mu_1}\left(\lambda + \mu_1 + \mu_2 - \sqrt{(\lambda-\mu_1)^2 + 2(\lambda+\mu_1)\mu_2 + \mu_2^2}\right). \tag{9}$$

Therefore, $\lim_{B\to\infty} S^{(B)} < \infty$ if and only if $\chi_1 < 1$. In that case,

$$\lim_{B\to\infty} S^{(B)} = \mathbf{e}^\top (I-M)^{-1}\mathbf{e}_1.$$

A straightforward calculation completes the proof of assertion (ii). $\square$

We note that $p_{(0,n)}^{(B)} = \frac{x_{B-n}^{(B)}}{S^{(B)}}$ and $p_{(1,n)}^{(B)} = \frac{y_{B-n}^{(B)}}{S^{(B)}}$ for $n = 0, 1, \ldots, B$. From this and (1) we have

$$p_{(0,B)}^{(B)} = \frac{1}{S^{(B)}}, \qquad p_{(1,B)}^{(B)} = 0. \tag{10}$$

Therefore, the following corollary is immediate from Lemma 2. This solves the first part of Conjecture 1 by Silva et al. [6].

Corollary 1. (i) *$p_{(0,B)}^{(B)}$ is strictly decreasing in B.*

(ii) *$\lim_{B\to\infty} p_{(0,B)}^{(B)} > 0$ if and only if $\chi_1 < 1$. In that case,*

$$\lim_{B\to\infty} p_{(0,B)}^{(B)} = \frac{\lambda(\mu_1-\mu_2) - \mu_1\mu_2}{\lambda\mu_1}.$$

In the remainder of this subsection, we find closed-form expressions for $p_{(0,n)}^{(B)}$ and $p_{(1,n)}^{(B)}$, $n = 0, 1, \ldots, B$. Note that for $i = 1, 2$, $\begin{bmatrix} \chi_i - \frac{\mu_2}{\mu_1} & \frac{\mu_2}{\mu_1} \end{bmatrix}^\top$ is an eigenvector of M corresponding to the eigenvalue χ_i. Thus M can be written as

$$M = \frac{\mu_1}{\mu_2(\chi_1-\chi_2)} \begin{bmatrix} \chi_1 - \frac{\mu_2}{\mu_1} & \chi_2 - \frac{\mu_2}{\mu_1} \\ \frac{\mu_2}{\mu_1} & \frac{\mu_2}{\mu_1} \end{bmatrix} \times \begin{bmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{bmatrix} \begin{bmatrix} \frac{\mu_2}{\mu_1} & \frac{\mu_2}{\mu_1} - \chi_2 \\ -\frac{\mu_2}{\mu_1} & \chi_1 - \frac{\mu_2}{\mu_1} \end{bmatrix}. \tag{11}$$

By (1), (2) and (11), we have that for $n = 1, \ldots, B$,

$$\begin{aligned} \begin{bmatrix} p_{(0,n)}^{(B)} \\ p_{(1,n)}^{(B)} \end{bmatrix} &= \frac{1}{S^{(B)}} \begin{bmatrix} x_{B-n}^{(B)} \\ y_{B-n}^{(B)} \end{bmatrix} \\ &= \frac{\mu_1}{\mu_2(\chi_1-\chi_2)S^{(B)}} \begin{bmatrix} \chi_1 - \frac{\mu_2}{\mu_1} & \chi_2 - \frac{\mu_2}{\mu_1} \\ \frac{\mu_2}{\mu_1} & \frac{\mu_2}{\mu_1} \end{bmatrix} \times \begin{bmatrix} \chi_1^{B-n} & 0 \\ 0 & \chi_2^{B-n} \end{bmatrix} \begin{bmatrix} \frac{\mu_2}{\mu_1} & \frac{\mu_2}{\mu_1} - \chi_2 \\ -\frac{\mu_2}{\mu_1} & \chi_1 - \frac{\mu_2}{\mu_1} \end{bmatrix} \mathbf{e}_1 \\ &= \frac{1}{(\chi_1-\chi_2)S^{(B)}} \begin{bmatrix} \chi_1^{B-n+1} - \chi_2^{B-n+1} - \frac{\mu_2}{\mu_1}(\chi_1^{B-n} - \chi_2^{B-n}) \\ \frac{\mu_2}{\mu_1}(\chi_1^{B-n} - \chi_2^{B-n}) \end{bmatrix}. \end{aligned} \tag{12}$$

By (3) and (12) with $n = 1$, we have

$$\begin{aligned} \begin{bmatrix} p_{(0,0)}^{(B)} \\ p_{(1,0)}^{(B)} \end{bmatrix} &= \begin{bmatrix} \frac{\mu_2}{\lambda} & 0 \\ \frac{\mu_2}{\mu_1} & \frac{\mu_2}{\mu_1} \end{bmatrix} \begin{bmatrix} p_{(0,1)}^{(B)} \\ p_{(1,1)}^{(B)} \end{bmatrix} \\ &= \frac{1}{(\chi_1-\chi_2)S^{(B)}} \begin{bmatrix} \frac{\mu_2}{\lambda}(\chi_1^B - \chi_2^B) - \frac{\mu_2^2}{\lambda\mu_1}(\chi_1^{B-1} - \chi_2^{B-1}) \\ \frac{\mu_2}{\mu_1}(\chi_1^B - \chi_2^B) \end{bmatrix}. \end{aligned} \tag{13}$$

Now it only remains to determine $S^{(B)}$. Since $\sum_{i=0}^{1}\sum_{n=0}^{B} p_{(i,n)}^{(B)} = 1$, we have by (12) and (13) that

$$\begin{aligned} S^{(B)} &= \frac{1}{\chi_1-\chi_2}\left(\sum_{k=1}^{B}\chi_1^k - \sum_{k=1}^{B}\chi_2^k + \left(\frac{\mu_2}{\mu_1} + \frac{\mu_2}{\lambda}\right) \times (\chi_1^B - \chi_2^B) - \frac{\mu_2^2}{\lambda\mu_1}(\chi_1^{B-1} - \chi_2^{B-1})\right) \\ &= \frac{1}{\chi_1-\chi_2}\left(\sum_{k=1}^{B+1}\chi_1^k - \sum_{k=1}^{B+1}\chi_2^k - \frac{\mu_2^2}{\lambda\mu_1}(\chi_1^B - \chi_2^B)\right), \end{aligned}$$

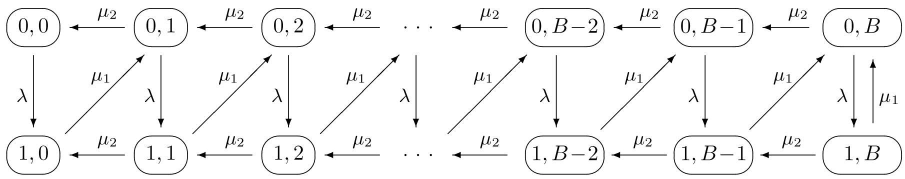

Fig. 2. The transition diagram of the Markov process $\{(\hat{N}_1(t), \hat{N}_2(t)) : t \geq 0\}$ under the greedy policy.

where the second equality follows from the characteristic equation of M, i.e., $\chi_i^2 - \left(\frac{\mu_2}{\mu_1} + \frac{(\mu_1+\mu_2)\mu_2}{\lambda\mu_1}\right)\chi_i + \frac{\mu_2^2}{\lambda\mu_1} = 0, i = 1, 2.$

In summary, we have the following result.

Theorem 1. *The stationary distribution for the system operating under the prudent policy is given by*

$$p_{(0,0)}^{(B)} = \frac{1}{(\chi_1 - \chi_2)S^{(B)}} \times \left(\frac{\mu_2}{\lambda}(\chi_1^B - \chi_2^B) - \frac{\mu_2^2}{\lambda\mu_1}(\chi_1^{B-1} - \chi_2^{B-1})\right),$$

$$p_{(0,n)}^{(B)} = \frac{1}{(\chi_1 - \chi_2)S^{(B)}} \times \left(\chi_1^{B-n+1} - \chi_2^{B-n+1} - \frac{\mu_2}{\mu_1}(\chi_1^{B-n} - \chi_2^{B-n})\right), \quad n = 1, \dots, B,$$

$$p_{(1,n)}^{(B)} = \frac{\mu_2}{\mu_1(\chi_1 - \chi_2)S^{(B)}}(\chi_1^{B-n} - \chi_2^{B-n}), \quad n = 0, 1, \dots, B,$$

where

$$S^{(B)} = \frac{1}{\chi_1 - \chi_2}\left(\sum_{k=1}^{B+1}\chi_1^k - \sum_{k=1}^{B+1}\chi_2^k - \frac{\mu_2^2}{\lambda\mu_1}(\chi_1^B - \chi_2^B)\right)$$

with χ_1 and χ_2 given in (8) *and* (9).

We remark that when $n = B$, Theorem 1 reduces to (10).

2.2. Greedy policy

Let $\hat{N}_i(t)$, $i = 1, 2$, denote the number of customers (including those waiting and in service) at station i at time t, in the system operating under the greedy policy. Then $\{(\hat{N}_1(t), \hat{N}_2(t)) : t \geq 0\}$ is a continuous time Matkov process with state space $\hat{\mathcal{S}} = \{(i, j) : i = 0, 1, j = 0, 1, \dots, B\}$. Fig. 2 depicts the state transition diagram of the Markov process $\{(\hat{N}_1(t), \hat{N}_2(t)) : t \geq 0\}$. As we can see from Figs. 1 and 2, all transition rates for the greedy policy are equal to the transition rates for the prudent policy except from state $(0, B)$ to $(1, B)$. Under the greedy policy the transition from state $(0, B)$ to $(1, B)$ is possible, whereas under the prudent policy the transition from $(0, B)$ to $(1, B)$ is not possible.

Let $\hat{p}_{(i,j)}^{(B)}$, $(i, j) \in \hat{\mathcal{S}}$ be the stationary distribution of the Markov process $\{(\hat{N}_1(t), \hat{N}_2(t)) : t \geq 0\}$, i.e.,

$$\hat{p}_{(i,j)}^{(B)} = \lim_{t\to\infty} \mathbb{P}((\hat{N}_1(t), \hat{N}_2(t)) = (i, j)).$$

Define

$$\hat{x}_n^{(B)} \equiv \frac{\hat{p}_{(0,B-n)}^{(B)}}{\hat{p}_{(0,B)}^{(B)} + \hat{p}_{(1,B)}^{(B)}}, \qquad \hat{y}_n^{(B)} \equiv \frac{\hat{p}_{(1,B-n)}^{(B)}}{\hat{p}_{(0,B)}^{(B)} + \hat{p}_{(1,B)}^{(B)}},$$

for $n = 0, 1, \dots, B$. As the procedure and arguments in this subsection are very similar to those in the previous subsection we omit the details of them.

Lemma 3. *We have the following recurrence formula:*

$$\begin{bmatrix} \hat{x}_0^{(B)} \\ \hat{y}_0^{(B)} \end{bmatrix} = \begin{bmatrix} \dfrac{\mu_1 + \mu_2}{\lambda + \mu_1 + \mu_2} \\ \dfrac{\lambda}{\lambda + \mu_1 + \mu_2} \end{bmatrix}, \tag{14}$$

$$\begin{bmatrix} \hat{x}_{n+1}^{(B)} \\ \hat{y}_{n+1}^{(B)} \end{bmatrix} = \begin{bmatrix} \dfrac{(\mu_1 + \mu_2)\mu_2}{\lambda\mu_1} & \dfrac{\mu_2^2}{\lambda\mu_1} \\ \dfrac{\mu_2}{\mu_1} & \dfrac{\mu_2}{\mu_1} \end{bmatrix} \begin{bmatrix} \hat{x}_n^{(B)} \\ \hat{y}_n^{(B)} \end{bmatrix}, \quad n = 0, 1, \dots, B-2,$$

$$\begin{bmatrix} \hat{x}_B^{(B)} \\ \hat{y}_B^{(B)} \end{bmatrix} = \begin{bmatrix} \dfrac{\mu_2}{\lambda} & 0 \\ \dfrac{\mu_2}{\mu_1} & \dfrac{\mu_2}{\mu_1} \end{bmatrix} \begin{bmatrix} \hat{x}_{B-1}^{(B)} \\ \hat{y}_{B-1}^{(B)} \end{bmatrix}.$$

Proof. By choosing $E = \{(1, B)\}$, we have from (4) that $\lambda\hat{x}_0^{(B)} = (\mu_1 + \mu_2)\hat{y}_0^{(B)}$. Combining this and $\hat{x}_0^{(B)} + \hat{y}_0^{(B)} = 1$, we get (14). The others follow by the same argument as in the proof of Lemma 1. □

Lemma 4. *Let $\hat{S}^{(B)} = \sum_{n=0}^{B}(\hat{x}_n^{(B)} + \hat{y}_n^{(B)})$. Then*

(i) *$\hat{S}^{(B)}$ is strictly increasing in B.*

(ii) *$\lim_{B\to\infty} \hat{S}^{(B)} < \infty$ if and only if $\chi_1 < 1$. In that case,*

$$\lim_{B\to\infty} \hat{S}^{(B)} = \frac{\lambda\mu_1(\lambda + \mu_1)}{(\lambda(\mu_1 - \mu_2) - \mu_1\mu_2)(\lambda + \mu_1 + \mu_2)}.$$

Proof. The proof is the same as that of Lemma 2, except the vector $\mathbf{e}_1$ is replaced by $\begin{bmatrix} \dfrac{\mu_1 + \mu_2}{\lambda + \mu_1 + \mu_2} & \dfrac{\lambda}{\lambda + \mu_1 + \mu_2} \end{bmatrix}^\top$. □

We note that $\hat{p}_{(0,n)}^{(B)} = \frac{\hat{x}_{B-n}^{(B)}}{\hat{S}^{(B)}}$ and $\hat{p}_{(1,n)}^{(B)} = \frac{\hat{y}_{B-n}^{(B)}}{\hat{S}^{(B)}}$ for $n = 0, 1, \dots, B$. From this and (14) we have

$$\hat{p}_{(0,B)}^{(B)} = \frac{\mu_1 + \mu_2}{\lambda + \mu_1 + \mu_2}\frac{1}{\hat{S}^{(B)}}, \qquad \hat{p}_{(1,B)}^{(B)} = \frac{\lambda}{\lambda + \mu_1 + \mu_2}\frac{1}{\hat{S}^{(B)}}. \tag{15}$$

Therefore, the following corollary is immediate from Lemma 4. This solves the second part of Conjecture 1 by Silva et al. [6].

Corollary 2. (i) *$\hat{p}_{(0,B)}^{(B)}$ and $\hat{p}_{(1,B)}^{(B)}$ are strictly decreasing in B.*

(ii) *$\lim_{B\to\infty} \hat{p}_{(0,B)}^{(B)} > 0$ if and only if $\chi_1 < 1$. Moreover, $\lim_{B\to\infty} \hat{p}_{(1,B)}^{(B)} > 0$ if and only if $\chi_1 < 1$. If $\chi_1 < 1$, then*

$$\lim_{B\to\infty} \hat{p}_{(0,B)}^{(B)} = \frac{\mu_1^2(\lambda - \mu_2) - \mu_2^2(\lambda + \mu_1)}{\lambda(\lambda + \mu_1)\mu_1},$$

$$\lim_{B\to\infty} \hat{p}_{(1,B)}^{(B)} = \frac{\lambda(\mu_1 - \mu_2) - \mu_1\mu_2}{(\lambda + \mu_1)\mu_1}.$$

Finally, we can obtain closed-form expressions for $\hat{p}_{(0,n)}^{(B)}$ and $\hat{p}_{(1,n)}^{(B)}$, $n = 0, 1, \dots, B$, by following the same procedure as in the derivation of Theorem 1 and using Lemma 3.

Theorem 2. *The stationary distribution for the system operating under the greedy policy is given by*

$$\begin{aligned}
\hat{p}^{(B)}_{(0,0)} &= \frac{1}{(\chi_1-\chi_2)\hat{S}^{(B)}}\Big\{\frac{\mu_2}{\lambda}(\chi_1^B-\chi_2^B) \\
&\quad -\frac{\mu_2^2}{\lambda}\left(\frac{1}{\mu_1}+\frac{1}{\lambda+\mu_1+\mu_2}\right)(\chi_1^{B-1}-\chi_2^{B-1}) \\
&\quad +\frac{\mu_2^3}{\lambda(\lambda+\mu_1+\mu_2)\mu_1}(\chi_1^{B-2}-\chi_2^{B-2})\Big\}, \\
\hat{p}^{(B)}_{(0,n)} &= \frac{1}{(\chi_1-\chi_2)\hat{S}^{(B)}}\Big\{\chi_1^{B-n+1}-\chi_2^{B-n+1} \\
&\quad -\mu_2\left(\frac{1}{\mu_1}+\frac{1}{\lambda+\mu_1+\mu_2}\right)(\chi_1^{B-n}-\chi_2^{B-n}) \\
&\quad +\frac{\mu_2^2}{(\lambda+\mu_1+\mu_2)\mu_1}(\chi_1^{B-n-1}-\chi_2^{B-n-1})\Big\}, \\
&\quad n=1,\ldots,B, \\
\hat{p}^{(B)}_{(1,n)} &= \frac{\mu_2}{\mu_1(\chi_1-\chi_2)\hat{S}^{(B)}}\Big\{\chi_1^{B-n}-\chi_2^{B-n}-\frac{\mu_2}{\lambda+\mu_1+\mu_2} \\
&\quad \times(\chi_1^{B-n-1}-\chi_2^{B-n-1})\Big\}, \quad n=0,1,\ldots,B,
\end{aligned}$$

where

$$\begin{aligned}
\hat{S}^{(B)} &= \frac{1}{\chi_1-\chi_2}\Big\{\sum_{k=1}^{B+1}\chi_1^k-\sum_{k=1}^{B+1}\chi_2^k \\
&\quad -\frac{\mu_2}{\lambda+\mu_1+\mu_2}\left(\sum_{k=0}^{B-1}\chi_1^k-\sum_{k=0}^{B-1}\chi_2^k\right) \\
&\quad -\mu_2\left(\frac{\mu_2}{\lambda\mu_1}+\frac{1}{\lambda+\mu_1+\mu_2}\right)(\chi_1^B-\chi_2^B) \\
&\quad +\frac{\mu_2^3}{\lambda(\lambda+\mu_1+\mu_2)\mu_1}(\chi_1^{B-1}-\chi_2^{B-1})\Big\}
\end{aligned}$$

with χ_1 and χ_2 given in (8) and (9).

We remark that when $n = B$, Theorem 2 reduces to (15).

3. Optimal policy

As mentioned before, Silva et al. [6] showed that either the prudent policy or the greedy policy can be optimal. Also, that if $c_1 \geq c_2$, then the greedy policy π_G is always optimal. They gave a criterion that determines which of the two policies is optimal for all cases including when $c_1 < c_2$. Let

$$c_*(B) = c_1\frac{\lambda}{\mu_1}\left(\frac{\sum_{i=0}^{B-1}\left(p^{(B)}_{(1,i)}-\hat{p}^{(B)}_{(1,i)}\right)+p^{(B)}_{(0,B)}}{\hat{p}^{(B)}_{(1,B)}}-1\right). \tag{16}$$

If $c_2 > c_*(B)$, then π_P is optimal for minimizing the long-run average cost; otherwise π_G is optimal, see Proposition 3 of [6]. They also provided a closed-form expression for $c_*(B)$ when $B \leq 10$ and illustrated through numerical examples that the threshold value for $B = 10$, $c_*(10)$, yields near optimal cost values even when $B > 10$.

Now we find a closed-form expression of $c_*(B)$ for any $B \geq 1$, by using the explicit closed-form expression for the stationary distribution given in Section 2. Recall that

$$p^{(B)}_{(0,B)} = \frac{1}{S^{(B)}}, \qquad \hat{p}^{(B)}_{(1,B)} = \frac{1}{\hat{S}^{(B)}}\frac{\lambda}{\lambda+\mu_1+\mu_2}. \tag{17}$$

For the greedy policy,

$$\sum_{i=0}^{B-1}\hat{p}^{(B)}_{(1,i)}+\hat{p}^{(B)}_{(1,B)} = \frac{\lambda}{\lambda+\mu_1},$$

the right-hand side of which is the probability that station 1 is busy. Thus,

$$\sum_{i=0}^{B-1}\hat{p}^{(B)}_{(1,i)} = \frac{\lambda}{\lambda+\mu_1}-\frac{1}{\hat{S}^{(B)}}\frac{\lambda}{\lambda+\mu_1+\mu_2}. \tag{18}$$

On the other hand,

$$\sum_{i=0}^{B-1}p^{(B)}_{(1,i)} = \frac{\mu_2}{\mu_1(\chi_1-\chi_2)S^{(B)}}\left(\sum_{k=1}^{B}\chi_1^k-\sum_{k=1}^{B}\chi_2^k\right). \tag{19}$$

Plugging (17)–(19) into (16) yields

$$\begin{aligned}
c_*(B) &= c_1\frac{(\lambda+\mu_1+\mu_2)\hat{S}^{(B)}}{\mu_1}\Big\{\frac{\mu_2}{\mu_1(\chi_1-\chi_2)S^{(B)}} \\
&\quad \times\left(\sum_{k=1}^{B}\chi_1^k-\sum_{k=1}^{B}\chi_2^k\right)+\frac{1}{S^{(B)}}-\frac{\lambda}{\lambda+\mu_1}\Big\}.
\end{aligned}$$

Therefore, we have the following theorem, which is an immediate consequence of Proposition 3 of [6]. The theorem allows us to characterize the optimal policy. The optimality condition is expressed in terms of the parameters of the model.

Theorem 3. *If*

$$\begin{aligned}
\frac{c_2}{c_1} &> \frac{(\lambda+\mu_1+\mu_2)\hat{S}^{(B)}}{\mu_1}\Big(\frac{\mu_2}{\mu_1(\chi_1-\chi_2)S^{(B)}} \\
&\quad \times\left(\sum_{k=1}^{B}\chi_1^k-\sum_{k=1}^{B}\chi_2^k\right)+\frac{1}{S^{(B)}}-\frac{\lambda}{\lambda+\mu_1}\Big),
\end{aligned} \tag{20}$$

then the prudent policy π_P is optimal; otherwise the greedy policy π_G is optimal.

Remark 1. Carefully examining the proofs of [6], we see that if $c_2 > c_*(B)$, then π_P is optimal but π_G is not optimal. If $c_2 = c_*(B)$, then both π_P and π_G are optimal. If $c_2 > c_*(B)$, then π_G is optimal but π_P is not optimal. Therefore, we have that if (20) holds, then π_P is optimal but π_G is not optimal. If $\frac{c_2}{c_1}$ equals the right-hand side of (20), then both π_P and π_G are optimal. If $\frac{c_2}{c_1}$ is strictly less than the right-hand side of (20), then π_G is optimal but π_P is not optimal.

Remark 2. From the result of [6] we know that if $c_1 \geq c_2$, then π_G is optimal, and, furthermore, π_P cannot be optimal. It follows from Theorem 3 that for any $\lambda > 0$, $\mu_1 > 0$, $\mu_2 > 0$, and $B \geq 1$, the right-hand side of (20) is strictly larger than 1.

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