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Recent advancements in robust optimization for investment management

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Abstract Robust optimization has become a widely implemented approach in investment management for incorporating uncertainty into financial models. The first applications were to asset allocation and equity portfolio construction. Significant advancements in robust portfolio optimization took place since it gained popularity almost two decades ago for improving classical models on portfolio optimization. Recently, studies applying the worst-case framework to bond portfolio construction, currency hedging, and option pricing have appeared in the practitioner-oriented literature. Our focus in this paper is on recent advancements to categorize robust optimization models into asset allocation at the asset class level and portfolio selection at the individual asset level, and we further separate robust portfolio selection approaches specific to each asset class. This organization provides a clear overview on how robust optimization is extensively implemented in investment management.

Keywords Robust optimization · Asset allocation · Portfolio selection · Investment management

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1 Introduction

Robust portfolio optimization has been one of the most implemented techniques in investment management for resolving the sensitivity of mean-variance portfolio optimization (Markowitz 1952). While mean-variance analysis requires mean, variance, and covariance of asset returns, the optimal allocation is highly dependent on the estimated value of these inputs. Because these values cannot be estimated with certainty, robust portfolio optimization models uncertainty in financial markets using uncertainty sets that contain possible realizations and the optimal portfolio in the worst case is computed for achieving robustness (Fabozzi et al. 2007). Achieving robustness aims at allowing certain degrees of uncertainty in inputs while generating manageable outputs. This goal is accomplished in robust optimization by defining uncertainty sets for representing possible input values and the robust solution is produced from worst-case or minimax approaches. Its main advantage over stochastic programming is less computational complexity because certain uncertainty sets result in robust counterparts of portfolio problems that are formulated as tractable optimization problems (Goldfarb and Iyengar 2003; Tütüncü and Koenig 2004). Even various extensions of the classical mean-variance model with risk measures such as value-at-risk and conditional value-at-risk can be reformulated as robust conic programming problems that are efficiently solved (El Ghaoui et al. 2003; Zhu and Fukushima 2009).

Naturally, many studies on robust optimization in portfolio management focus on various formulations of portfolio problems, selection of appropriate uncertainty sets, and derivation of robust counterparts that lead to tractable robust models (Fabozzi et al. 2010; Kim et al. 2014b). Earlier work on robust approaches especially applies the worst-case technique under a simple investment setting where multiple risky assets are considered without distinguishing different asset types. Recently, an increasing number of studies have addressed ambiguity in different stages of the portfolio management process as well as uncertainty specific to certain asset classes. Robust optimization is no longer only applied to asset allocation or stock portfolio construction; the worst-case framework is also used in areas of investment management such as hedging currency risk and pricing derivatives.

Therefore, in this paper, we survey recent developments in robust optimization from an investment management perspective in order to provide an overview of how robust models are used in various domains of investment management. This is a notable distinction from other papers in the literature that outline robust portfolio optimization, which have been mostly organized based on problem formulation such as the complexity of the portfolio problem, the type of uncertainty set, and the risk measure employed. More importantly, there have been many studies on robust investment management in recent years and, in order to present an up-to-date overview, we primarily concentrate on developments since 2010 (with only a few exceptions) that have sparked research in new applications. We purposely omit detailed formulations in this paper because our objective is to explain which components are modeled as uncertain, why robust optimization is applied in each case, and how robust optimization improves risk management in different stages of the investment management process.

The organization of the paper is as follows. We follow the portfolio management process and, thus, first discuss asset allocation and then cover individual asset classes separately. Sect. 2 begins by demonstrating the sensitivity of the classical asset allocation approach before reviewing studies on robust asset allocation and robust asset-liability management. In Sect. 3, robust optimization for equity portfolios, bond portfolios, and currency portfolios are examined. Worst-case approaches for using derivatives in portfolio management and for option pricing are presented in Sect. 4. Section 5 concludes the paper.

2 Asset allocation

In the investment management process, asset allocation is where the first set of allocation decisions is made (Maginn et al. 2007). Asset allocation greatly affects portfolio behavior (Brinson et al. 1995), as well as being the critical step in selecting the list of candidate asset classes. As the traditional approach to asset allocation is the mean-variance framework, much of the advancement on robust asset allocation is various reformulations of the classical mean-variance problem. We begin this section by demonstrating the sensitivity of the traditional approach and then present the robust models. In addition, asset-liability management problems that utilize the worst-case framework are introduced.[1]

2.1 Sensitivity of mean-variance asset allocation

In finance, robust optimization has been applied most frequently in portfolio allocation due to the optimal portfolio weights of the mean-variance framework being highly dependent on the estimated input values. This concern has been widely observed through comprehensive empirical analyses. The high sensitivity in portfolio weights caused by perturbation in the estimation of a single asset is commented in Best and Grauer (1991a, b). In Michaud (1989), the mean-variance model is referred to as estimation-error maximizers because the overweighted assets tend to be the ones with the largest estimation errors. Similar conclusions on the impact of input sensitivity on optimal portfolio weights are reported by Broadie (1993), Chopra and Ziemba (1993), among others.

Since the main focus of this paper is the utilization of robust methods in investment management, we demonstrate its essential need with a simple example on how allocations can be easily shifted, and consequently affecting the portfolio's risk exposure. In our experiment, an asset manager uses the classical mean-variance optimization for allocation among four asset classes where the mean vector and covariance matrix of returns are estimated from the weekly returns during the first 13 weeks (i.e., 3 months) of 2016. In particular, the formulation that minimizes portfolio variance with a certain level of portfolio return is optimized where negative allocations (i.e., short selling) are avoided. It is then observed how the optimal allocation changes when one additional observation (i.e., return on the 14th week of 2016) is included for estimating the expected return of a single asset class. Consideration of including one additional observation for estimating expected returns is a decision that routinely arises in portfolio management. But as shown next, this rather trivial choice can lead to significant changes in the optimal portfolio weights if the uncertainty in returns is not modeled. Four major U.S. asset classes are considered with the returns of the representative indices collected from Datastream: stocks (Russell 3000), bonds (Datastream U.S. government bond index), commodity (DJ-UBS commodity index), and real estate (MSCI US REIT).

As shown in Fig. 1, three separate comparisons are presented for portfolios with weekly return levels of 0.1, 0.2, and 0.25%. In each plot for a single asset class, case 0 is when 13-week returns are used for all estimations, case 1 is identical to case 0 except only the expected return of stocks is estimated from 14-week returns, case 2 is identical to case 0 except only the expected return of bonds is estimated from 14-week returns, and so on. For example, the plot for stocks in panel A of Fig. 1 shows that about 50% is invested in stocks when the 13-week returns are used, but the allocation to stocks decreases to less than 10% when the expected return of commodities is estimated from the 14-week returns. This is a significant shift in the allocation to stocks caused by a minor difference in how the expected return of

[1] A survey of robust asset allocation is also presented in Scutellà and Recchia (2013).

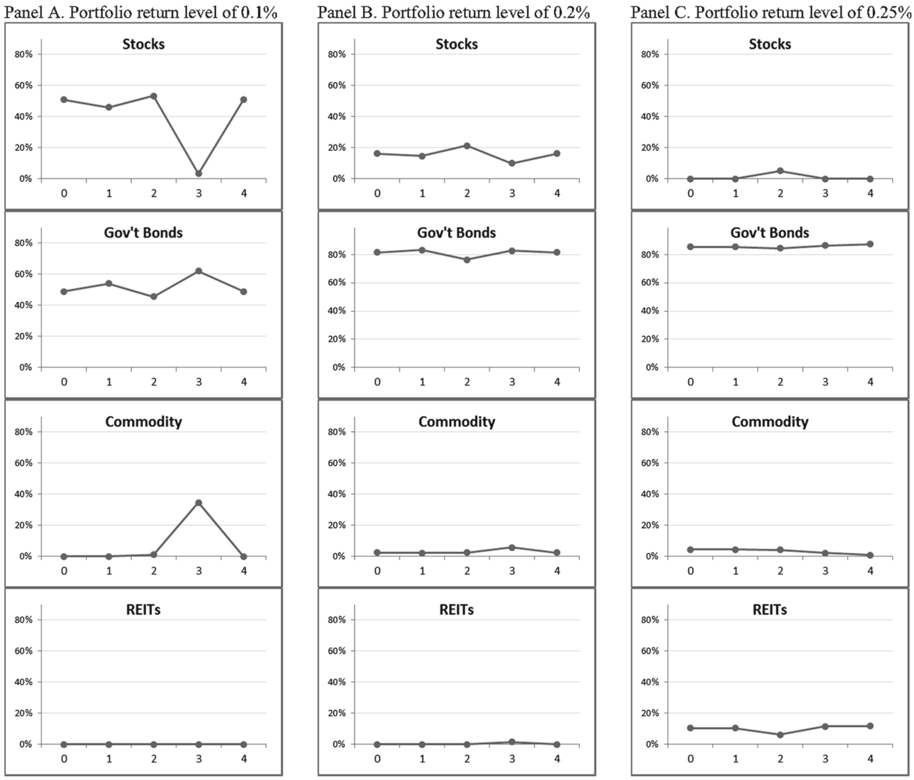

Fig. 1 Portfolio allocation for three separate weekly portfolio return levels (non-negative weights)

a single asset class is computed. Although the other outcomes might not seem like critical adjustments, stocks will not be considered attractive in all cases unless government bond returns are estimated from the first 14 weeks of 2016. The deviation is much more dynamic when short positions are allowed, as shown in Fig. 2. It is clear from these simple examples that robustness is essential in portfolio risk management mainly because of ambiguity in asset returns.[2]

2.2 Robust asset allocation

One of the first robust approaches to asset allocation was introduced by Tütüncü and Koenig (2004). They present robust formulations when the expected return vector and the covariance matrix of asset returns are defined by lower and upper bounds and also illustrate how to compute the robust efficient frontier. Uncertainty sets based on bootstrapping and moving averages are explored and numerical experiments with various asset classes demonstrate how robust portfolios have improved worst-case performance, stability over time, and concentration on a small set of asset classes. Recently, a variation of this robust model is introduced by Yam et al. (2016) that allows short positions. Investigating several robust formulations of

[2] In both Figs. 1 and 2, the allocation to stocks decreases whereas the allocation to bonds increases as the portfolio return level is raised. This is observed because bonds have relatively high expected return with low variance and stocks have negative expected return with high variance during the first 3 months of 2016.

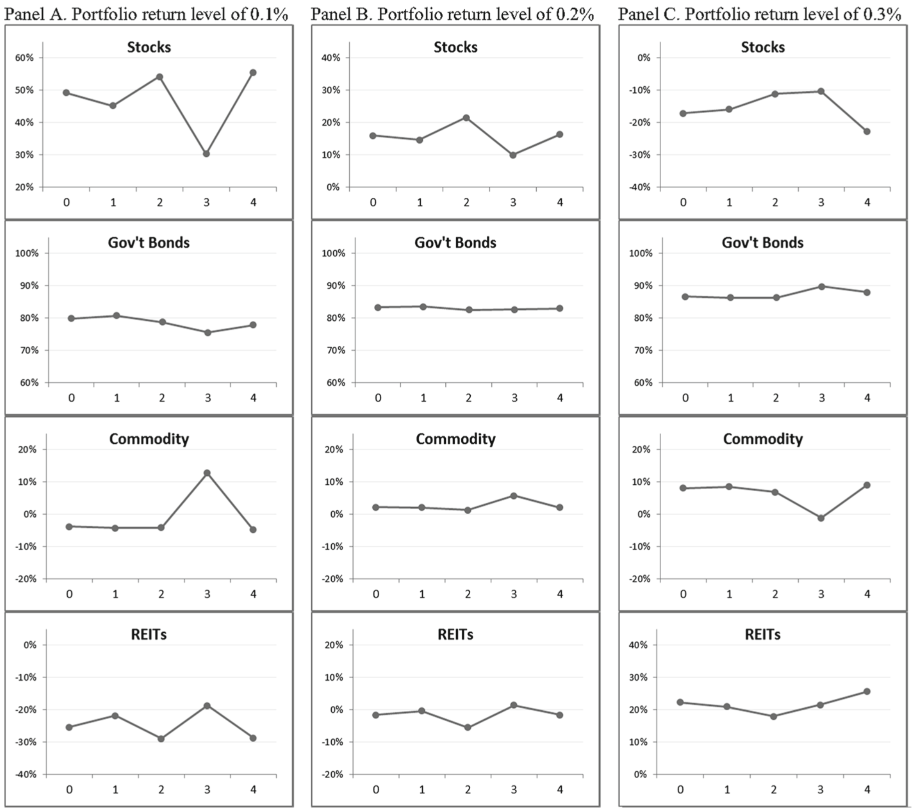

Fig. 2 Portfolio allocation for three separate weekly portfolio return levels (shorting allowed)

mean-variance problems, they find that the effect of uncertainty in expected returns is more critical than the uncertainty in covariance matrix for controlling sensitivity.

The practical advantage of applying robust models to strategic asset allocation is demonstrated by Asl and Etula (2012). In their robust asset allocation approach, they perform robust optimization with input estimated from a multi-factor model, which is suitable for estimating expected returns and risk across asset classes. Their empirical tests reveal less dispersion in robust portfolios invested in 15 asset classes and, thus, considered to be more suited for asset allocation than classical approaches.

The out-of-sample performance of robust portfolios is tested by Ben-Tal et al. (2010), where they introduce a soft robust approach that relaxes the robustness of the standard robust optimization models. They demonstrate its benefits by comparing performance of portfolios that invest in 11 asset classes. The out-of-sample results demonstrate a higher gain for the soft robust approach while only sacrificing limited downside risk. Recchia and Scutellà (2014) compare performances of several robust asset allocation strategies and confirm empirically that the relaxed robust models of Ben-Tal et al. (2010) show strong robustness. Moreover, they find that the classical robust model of Tütüncü and Koenig (2004), which minimizes worst-case variance, exhibits low turnover and high diversification.

Although some studies are not only targeting allocation among asset classes, they nonetheless make great contribution to asset allocation by considering more practical investment situations. For example, constraining the number of assets in a portfolio is desired in asset allocation. Sadjadi et al. (2012) derive robust formulations of portfolio optimization problems for constructing portfolios with a predefined limited number of assets when asset returns are subject to uncertainty and also develop a procedure based on genetic algorithms for finding the optimal robust portfolio. They also demonstrate its implementation with historical data for the Hang Seng, DAX 100, FTSE 100, S&P 100, and Nikkei 225.

Similarly, Gülpınar et al. (2011) analyze robust counterparts of a portfolio problem with discrete asset choice constraints that control the cardinality and buy-in threshold of a portfolio. Although the side effect of including these constraints that reflect investors' asset choice preferences in practice is the discontinuity of efficient frontiers, robust portfolio optimization with ellipsoidal uncertainty sets are shown to reduce the discontinuity caused by these constraints. They also demonstrate the advantage of robust portfolios by comparing the actual efficient frontiers, which reflect actual investing situations where portfolios are optimized from estimated values and expected portfolio returns are computed from true values.

2.3 Application in asset-liability management

Asset-liability management (ALM) mainly applies to pension plans with long investment horizons that need to manage assets relative to projected liabilities. Although ALM problems are traditionally solved using stochastic programming, several studies utilize robust optimization because of its computational advantage.

Iyengar and Ma (2010) consider an ALM problem for defined-benefit pension plans. The pension fund management problem is formulated as a chance-constrained optimization problem and a robust problem is created with a robust constraint that replaces the chance constraint, which becomes a second-order cone programming problem. Furthermore, they demonstrate how the robust approach leads to more conservative but robust performance, and conclude that robust methods are well suited for large-scale pension fund management problems because of their reduced computational complexity compared to stochastic programming.

Gülpınar and Pachamanova (2013) present an ALM model for pension funds based on robust optimization with ellipsoidal uncertainty sets and also discuss formulations for modeling time-varying investment opportunities with a vector-autoregressive process. In their model, uncertainty exists in asset returns and interest rates, which determine the asset value and the value of future liabilities, respectively. Although their empirical findings do not provide a conclusive comparison between the robust ALM approach and classical stochastic programming methods, they do highlight the computational advantage of the robust model.

A similar approach is taken by Gülpınar et al. (2016) for solving an ALM problem for a company that issues investment products with guarantees, where the two uncertain components are asset returns and the value of future liabilities. They propose symmetric and asymmetric uncertainty sets and present simulation results to show how robust approaches perform well under uncertain situations with fast computation time.

2.4 General robust allocation models

In Sect. 2.4, we mention notable studies that may not be classified as robust asset allocation or robust ALM but, nonetheless, present strategies that improve robustness when managing portfolios.

Sharpe ratio is one of the key measures of risk-adjusted performance and the portfolio problem of maximizing Sharpe ratio under asset return uncertainty was one of the earlier applications of robust portfolio optimization (Goldfarb and Iyengar 2003; Tütüncü and Koenig 2004). Deng et al. (2013) also optimize the maximization of worst-case Sharpe ratio, but use uncertainty sets of Sharpe ratio estimators. Their model selects the portfolio with the largest worst-case Sharpe ratio within a given confidence interval, which is shown to be equivalent to maximizing the value-at-risk-adjusted Sharpe ratio based on the observation that Sharpe ratio estimators are normally distributed.

Unlike the popular approach for considering the tradeoff between risk and return, Fliege and Werner (2014) take the multi-objective approach for robust asset allocation. They derive the robust counterpart to a multi-objective programming problem where the two objectives are minimizing portfolio risk and maximizing portfolio return. They also demonstrate how robust efficient frontiers can be used for finding an appropriate allocation under uncertainty.

Instead of directly assuming ambiguity in asset returns, Lutgens and Schotman (2010) model uncertainty through advice obtained from multiple expert advisors on mean vector and covariance matrices. The robust portfolio is found by evaluating the worst case among multiple recommendations. They analyze two situations where ambiguity in advice exists either only in expected returns or in both the mean and covariance matrix. The benefits from considering a robust approach based on multiple recommendations are stressed, especially when advisor recommendations are dispersed.

An alternative to solving mean-variance portfolio optimization is to consider the Kelly criterion (Kelly 1956). The strength of growth-optimal portfolios for a long period may be promising, but the strategy reveals volatile short-term movements and it requires complete information of the return distribution. Rujeerapaiboon et al. (2015) propose a robust growth-optimal model for finite investment periods and uncertain return distributions: the worst-case value-at-risk of the portfolio growth rate is maximized for forming robust optimal portfolios. Empirical results show that this robust model outperforms the classical growth-optimal portfolio as well as several classical mean-variance portfolios.

3 Portfolio selection

In portfolio management, once allocations among asset classes are determined, the portfolio is further optimized within each asset class. Since different asset classes have distinct characteristics, robust models should be developed independently for each asset class to effectively manage the relevant risk of the asset class. Here, we discuss robust approaches for equity, fixed-income, and currency markets. We also note that robust optimization is also applied to optimal portfolio execution. As execution is also an important stage in portfolio management, the optimal liquidation strategy for a portfolio can be strengthened through a minimax approach (Moazeni et al. 2013).

3.1 Equity markets

A good number of advances in robust portfolio optimization study models for stock portfolios, especially because of the high volatility in stock markets that call for robust approaches.[3] In particular, there have been many attempts to accurately model uncertainty in stocks by incorporating attributes of stock returns such as skewness and fat tails for forming robust stock portfolios.

[3] Kim et al. (2016b) provide a detailed explanation on how to formulate robust stock portfolio problems and also a step-by-step guide on finding optimal robust stock portfolios using MATLAB.

For example, Kawas and Thiele (2011) introduce a log-robust optimization model based on the log-normal behavior of stock prices. While their model is based on the traditional log-normal model, the worst-case approach of robust optimization takes into account the fat-tail events that are under-represented in the traditional approach. Their model also utilizes the price-of-robustness approach of Bertsimas and Sim (2004) for controlling the conservativeness but it is formulated as a linear program that can be efficiently solved. The log-robust approach is shown to form portfolios that are more diversified with better value-at-risk performance compared to traditional robust portfolio approaches. As an extension, Pae and Sabbaghi (2014) develop a log-robust optimization with transaction costs. The optimal portfolio is calculated using linear approximation and the portfolio is confirmed to outperform the model without transaction costs (Kawas and Thiele 2011) when large transaction volume is required.

The asymmetry of stock returns and increased correlation during stock market downturns are addressed by Kim et al. (2015). They demonstrate the value of worst-case information in the stock market for gaining robust performance by introducing a simple rule-based approach that focuses on worst-case returns for constructing robust portfolios. They discuss the advantage of focusing on worst-case information by relating the approach to robust portfolio optimization. Thus, Kim et al. (2015) provide support based on stock market behavior for applying worst-case optimization to portfolio management.

Also addressing return asymmetry, Chen and Tan (2009) introduce a robust optimization formulation that is suitable for reflecting the asymmetric returns in equity markets. In particular, they develop interval random uncertainty set, which is an interval set with randomly fluctuating bounds where upside and downside deviations are modeled separately. The mean-variance portfolio optimization problem is reformulated by applying interval random chance-constrained programming and the robust model is empirically shown to be more effective when investing in small-cap stocks.

Performance of robust stock portfolios is examined by Guastaroba et al. (2011). They investigate the in-sample and out-of-sample performances in the stock market of two robust optimization formulations with conditional value-at-risk as the risk measure. Using the 100 stocks comprising the FTSE 100 index as their universe of candidate stocks, they find that robust techniques perform well during downward periods but the advantage was not clear in other cases.

Robust stock portfolio performance is further investigated by Fastrich and Winker (2012) under a more realistic trading situation that only allows discrete portfolio weights to eliminate purchasing fractional shares, limits the maximum number of assets held in a portfolio, prohibits short-selling, and takes into account transaction costs. They suggest a hybrid heuristic algorithm to solve the robust portfolio problem under these settings and compare robust models of Tütüncü and Koenig (2004), Ceria and Stubbs (2006), and an extension of the Ceria and Stubbs model that includes uncertainty in the covariance matrix. By performing empirical tests with stocks in the DAX 100 index, they conclude that robust formulations lead to superior performance, especially when incorporating uncertainty in both the mean vector and covariance matrix.

Instead of proposing robust formulations or comparing performance, some studies have analyzed attributes of robust portfolios, finding that robust models with ambiguity in expected stock returns form portfolios that have higher factor exposure (Kim et al. 2013, 2014a, 2016a). Robust portfolios are shown to have a higher dependency on factor returns through the Fama-French three factor model and principal components analysis.

Although most of the above studies on robust stock portfolios aim for active returns while controlling risk, robust portfolio optimization can also be used for constructing a portfolio for index tracking. Chen and Kwon (2012) develop a robust model for tracking a market index. Instead of explicitly limiting the tracking error of a portfolio, the model maximizes pairwise similarities between assets of the portfolio and the target index, and the robustness is increased by modeling the uncertainty in similarities through a range of possible values. The proposed robust portfolio model allows setting the number of allowed assets and the model is formulated as a 0–1 integer program for identifying the assets included in the tracking portfolio. The advantage of the robust index tracking problem is demonstrated through portfolios that track the S&P 100 index.

A robust approach that avoids the use of uncertainty sets on expected returns is also introduced by Nguyen and Lo (2012). Ranking models have been used in portfolio selection in order to avoid estimating expected returns, and Nguyen and Lo (2012) derive a robust ranking problem that can be applied to portfolio optimization when there is uncertainty in the rankings. Testing the post-earnings-announcement drift effect with stocks in the Dow Jones Industrial Average Index, they confirm that the proposed robust ranking model reduces portfolio risk.

3.2 Credit and bond markets

Unlike stocks, the main uncertainty in fixed-income securities is related to the occurrence of defaults. A number of studies apply robust optimization for forming bond portfolios. For example, Ben-Tal et al. (2010) illustrate their soft robust approach with a bond portfolio that invests in 49 bonds in addition to one risk-free asset. The comparison suggests that the soft robust approach reduces the high conservatism of the standard robust optimization approach, which is shown to invest heavily in the risk-free asset.

Several studies propose robust models that are suitable for reducing risk specific to fixed-income securities. Shen et al. (2014) address long-term investment planning of insurance companies and pension funds and the challenges that arise in long-dated liability valuation. They consider investment in bonds for minimizing the expected shortfall of long maturity commitments where a minimax formulation for the optimal bond portfolio allocation models uncertainty in the risk premium of long-term bonds.

The impact of credit risk model misspecification is critical for bond portfolios because estimation of default intensity is difficult due to the rarity of default events. Bo and Capponi (2016) develop a dynamic robust bond portfolio model for investing in risky bonds that is robust against misspecifications of the reference credit model, deriving Hamilton-Jacob-Bellman equations for solving their robust problem.

Examination of defaults is also performed by Jaimungal and Sigloch (2012), who consider a hybrid credit model combining two main approaches for modeling default events, which are intensity-based and structural approaches. They apply the worst-case approach to value credit derivatives on a firm and derive robust indifference yields and credit default swap spreads. They find that including ambiguity aversion helps estimating yield curves and credit default swap spreads, being especially valuable for explaining short-term spreads.

3.3 Currency markets

While the most popular approach for reducing foreign exchange rate risk is hedging strategies with derivatives, there have been studies that apply robust optimization to portfolios that are exposed to currency risk.

Rustem and Howe (2002) explain worst-case approaches for managing pure currency portfolios composed of multiple currencies. They suggest a robust formulation for a currency portfolio optimization problem where the uncertainty in future exchange rates is modeled by setting upper and lower bounds for each currency. In addition to considering the worst-case currency returns in their formulation, they model the triangular relationship among exchange rates to avoid cross-currency inconsistency. They begin with the classical mean-variance model that finds the tradeoff between return and risk of currency portfolios, but present various minimax formulations for hedging currency exposure of general asset portfolios as well as currency portfolios with transaction costs where uncertainty sets are again defined by individual ranges.

The work of Rustem and Howe (2002) is extended by Fonseca et al. (2011), which presents an approach that applies robust optimization when allocating a portfolio among several currencies where the uncertainty set is defined as an ellipsoid. They also model the triangular relationship among exchange rates to bind the worst case within no arbitrage conditions. Furthermore, they demonstrate how currency options can be included in order to also insure cases when the worst case is realized outside of the uncertainty set defined in the robust optimization formulation. The main advantage of applying robust optimization is the increased flexibility compared to standard hedging strategies that only utilize derivatives such as forwards or futures.

A similar analysis is performed by Fonseca et al. (2012); they also apply robust optimization to international portfolios and also protect against cases outside of the defined uncertainty set with options. However, instead of currency portfolios, they analyze portfolios investing in assets denominated in different currencies and, thus, are exposed to currency risk. Although uncertainty in their model exists in both asset and currency returns, they derive a semidefinite programming approximation of their robust formulation. Moreover, the triangular relationship of foreign exchange rates is also reflected in their model similar to Fonseca et al. (2011) and demonstrates the flexibility of their proposed robust model.

Fonseca and Rustem (2012b) continue the work of Fonseca et al. (2012) on reducing ambiguity in asset and currency returns of international portfolios. They revisit the conventional approach of using forward rates to hedge against currency risk and show how forward contracts can be incorporated into worst-case formulations for international portfolios. Furthermore, forming robust international portfolios under uncertainty in returns of assets and currencies in a multi-period setting is studied by Fonseca and Rustem (2012a). A comparison between a single stage model and their multi-stage model is included to demonstrate the benefits of their approach.

4 Derivatives

Injecting portfolios with options for hedging purposes is a common strategy in controlling portfolio risk. Since the payoffs of options are driven by the underlying assets, uncertainty in the returns of underlying assets affects not only allocation in those assets but options as well. Thus, robust optimization has been applied to model the uncertainty in asset returns for portfolios including risky assets as well as options on those assets. In these portfolios, options may provide protection when asset returns are realized outside of uncertainty sets. Also, robust replicating portfolios are used for pricing derivatives as we discuss towards the end of this section.

4.1 Portfolios containing derivatives

Earlier models on derivative portfolios consider European style options where the investment period ends when the option expires. Lutgens et al. (2006) use robust optimization for hedging with options. They begin by deriving the expected return of options in terms of the possible realized returns of the underlying stocks. Their approach models uncertainty in the expected returns of the underlying stocks since option returns are driven by the stock returns. Investment in risky assets and options on these assets are designed by including robust constraints on portfolio performance. In order to demonstrate robust hedging, the robust technique is applied to several hedging scenarios, where investment in options on an index is combined with the constituent stocks of the index as well as options on those stocks.

Similar to Lutgens et al. (2006), Zymler et al. (2011) also study robust portfolios that invest in both stocks and their options in a single-period setting. While Zymler et al. also adopt ellipsoidal uncertainty sets, they present a variation of the portfolio optimization problem for stocks and options that has a computational advantage. They also formulate a robust portfolio optimization model that insures cases where realized returns are not from the uncertainty set. Empirical analyses from simulated and historical data confirm the better out-of-sample performance of their robust approaches compared to classical mean-variance portfolios. In particular, the insured robust model is shown to be more effective during extreme events.

Robust equity portfolios with option protection are also studied by Ling and Xu (2012). Their model is an extension of Goldfarb and Iyengar (2003), which assumes a factor model for modeling uncertainty in returns that defines a joint uncertainty set. Nonetheless, the objective of designing a robust portfolio with options is explored to hedge risk from extreme events that are not covered by uncertainty sets and the advantage is illustrated using simulated data and returns from the Chinese stock market.

Whereas the above studies assume a single-period investment where the option expires at the end of the period, Marzban et al. (2015) address multi-period portfolio selection when American style options are available. In this case, the exercise time of options also becomes a decision variable and the wealth at each period can be expressed by the possible actions an investor can take. They derive the multi-period problem for maximizing terminal wealth, which is referred to as the insured multi-period robust portfolio optimization problem. Optimal allocation among 30 stocks and 40 options on each stock is analyzed for a 3-month period and improvements are demonstrated for data with high and low levels of error.

Application of robust optimization to hedging barrier options is investigated by Maruhn and Sachs (2009). Hedge portfolios consisting of bonds and tradable call options for barrier option replication are exposed to changes in the volatility surface, and a robust static hedging strategy is developed to ensure that the value of the hedge portfolio is no lower than the payoff of the barrier option.

Risk management in exotic derivatives can also benefit from the worst-case framework. Gülpınar and Çanakoḡlu (2016) propose robust approaches for managing portfolios exposed to temperature uncertainty. Weather derivatives have been used for hedging risk by property and casualty insurance companies, hedge funds, and energy companies. Gülpınar and Çanakoḡlu derive the robust counterpart of a portfolio problem for weather futures contracts on temperature indices where temperature uncertainty is modeled by ellipsoidal or asymmetric uncertainty sets. While empirical analyses reveal better worst-case performance and diversification for the robust approaches, Gülpınar and Çanakoḡlu stress the importance of defining suitable uncertainty sets.

In addition, as already discussed in Sect. 3.3, notable work on robust currency portfolios also design robust problems with derivatives for hedging purposes.

4.2 Options pricing

Portfolio optimization can be used for option pricing through a replicating portfolio where the portfolio matches the payoff of an option. Similar to previous accounts on robust portfolio optimization, minimax methods can be applied for constructing replicating portfolios with reduced sensitivity.

Robust option pricing is studied by Bandi and Bertsimas (2014), who combine the replicating strategy with robust portfolio optimization. In particular, the worst-case replication error of a replicating portfolio is minimized, which consists of stocks and a risk-free asset. The uncertainty in the underlying price dynamics is modeled using polyhedral sets and this results in robust option pricing problems that are linear programs. Furthermore, they suggest robust option pricing for various options such as American options, Asian options, barrier options, and high-dimensional options that depend on a large number of underlying assets.

Finally, DeMarzo et al. (2016) also present an approach for robust option pricing but they focus on calculating robust bounds for option prices. The proposed robust bounds on price paths based on regret minimization is valuable because the bounds only depend on realized quadratic variation of the price process and are independent of the specific price process, the timing of trade, and the underlying price kernel.

5 Discussion on optimization methods

As summarized in previous sections, robust optimization is applied to a wide range of investment situations. Robust optimization problems take various forms depending on how the uncertain components are modeled. Furthermore, the resulting robust formulations may require relaxation or heuristic algorithms for efficiently solving for the optimal robust investment. Therefore, in this section, we summarize the robust models with focus on optimization methods. While a description of robust studies is included in the earlier sections, this section provides an overview.

Since robust optimization introduces an uncertainty set and formulates the robust counterpart of the original investment problem for computing the worst-case optimal solution, the resulting robust problem has a different formulation and often with higher complexity. Thus, it is most important to analyze the type of uncertainty sets and the formulation of robust counterparts. In fact, the choice of uncertainty sets determines whether the resulting robust problems can be reformulated as either linear programs, quadratic programs, or other optimization problems and, hence, also controls whether there exists known efficiently algorithms for solving the resulting robust formulations. This is summarized in Table 1 to show the popular choice of uncertainty sets and corresponding types of robust formulations for different investment problems.

Overall, the focus of uncertainty is on in mean, variance, and covariance of returns. Portfolio robustness can be increased by modeling uncertainty in the returns of the candidate assets and this is the approach taken by most robust models for asset allocation, asset-liability management, equity investments, and currency portfolios. Since derivative securities depend on underlying asset movements, many robust optimization approaches for derivative investments address uncertainty in the underlying such as stock returns or temperature.

In terms of the geometry of uncertainty sets, interval and ellipsoidal sets are most widely used. The worst case of interval uncertainty sets can usually be expressed as linear constraints and can be included without affecting the complexity of the investment problem. Definitions

Table 1 Uncertainty sets and optimization models used in investment management

	Uncertain components	Uncertainty sets	Optimization models
Asset allocation	Mean, variance, and covariance of asset returns	Interval, ellipsoidal, norm	Second-order cone programming, mixed-integer programming, multi-objective programming
Asset-liability management	Mean of asset returns	Ellipsoidal	Second-order cone programming
Equity portfolios	Mean, variance, and covariance of stock returns	Interval, discrete	Linear programming, mixed-integer programming
Currency portfolios	Currency returns	Interval, ellipsoidal, joint	Semi-definite programming
Derivatives	Mean, variance, and covariance of underlying stock returns	Interval, ellipsoidal, joint, norm	Linear programming, second-order cone programming, semi-infinite programming

Table 2 Various problem formulations of robust optimization

Problem formulations		Example articles
Optimization model	Linear programming	Kawas and Thiele (2011)
	Mixed-integer programming	Gülpınar et al. (2011), Chen and Kwon (2012), Nguyen and Lo (2012)
	Second-order cone programming	Gülpınar and Pachamanova (2013), Lutgens et al. (2006), Zymler et al. (2011)
	Semi-definite programming	Fonseca et al. (2012), Rujeerapaiboon et al. (2015)
	Multi-objective programming	Fliege and Werner (2014)
Problem type	Multi-stage problem	Fonseca and Rustem (2012a), Marzban et al. (2015)
	Regret minimization	DeMarzo et al. (2016), Xidonas et al. (2017)
	Ranking problem	Nguyen and Lo (2012)
Uncertainty set	Interval	Tütüncü and Koenig (2004), Rustem and Howe (2002)
	Ellipsoidal	Fonseca et al. (2011), Gülpınar et al. (2016)
	Joint	Ling and Xu (2012), Fonseca and Rustem (2012b)
	Norm, Polyhedral	Bandi and Bertsimas (2014), Sadjadi et al. (2012)
Constraint	Chance-constrained	Iyengar and Ma (2010), Rujeerapaiboon et al. (2015)
	Cardinality-constrained	Gülpınar et al. (2011), Chen and Kwon (2012)

based on d-norms introduced by Bertsimas et al. (2004) also lead to polyhedral uncertainty sets. On the other hand, ellipsoidal uncertainty sets lead to adding a second-order cone constraint to the original problem.

Furthermore, the worst case approach of robust optimization applied to problems with cardinality constraints results in mixed-integer programming. While the majority of the studies focus on robust counterparts of single-period models, robust optimization can be applied to decisions each period that result in robust multi-stage optimization. Various characteristics of robust optimization formulations for investment management are also summarized in Table 2 along with a few notable articles.

6 Conclusion

Robust optimization is a worst-case approach that is effective in modeling uncertain situations and it is widely explored in portfolio management. An increasing number of investment problems are applying worst-case models as a solution to dealing with uncertain movements of financial instruments. As we have discussed in this paper, the flexibility of the robust optimization framework allows for application to various investment problems. The latest developments on robust optimization in investment management address a wide range of problems, including asset-liability management and pricing derivatives, and a variety of asset classes, including bond and currency markets.

More recently, robust models have also been applied to topics of practical interest such as risk budgeting (Kapsos et al. 2017), factor-based investing (Kim et al. 2017), and incorporating investor views (Hasuike and Mehlawat 2017). As mentioned above, the worst-case approach can be applied to practical investment settings due to the flexibility of the approach. More importantly, robust optimization becomes a powerful strategy in practice because robustness is achieved without heavily penalizing the computational complexity of the problem. We believe the use of worst-case approaches for managing risk in financial modeling will continue to grow because of the applicability of robust optimization and also since uncertain behavior is accepted as an inherent characteristic of investing in financial markets.

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