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# Optimal investment risks and debt management with backup security in a financial crisis

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ABSTRACT

This paper examines a theoretical and an empirical study of an optimal investment management strategies and debt profile of an investor in a financial crisis. In order to minimize the incident of credit risks, the debts are backup with collaterals. The investment strategies and consumption plan of an investor are exposed to diffusion and credit risks. In this paper, we put into consideration a market that is exposed to four background risks which include inflation, investment, fixed asset and income risks. The investor's income process is influenced by the impact of labor force with the production rate function and is assumed to be stochastic. The market is categorized into two: financial market (FM) and fixed asset (FA) market. The underlying assets in the FM are stocks and a riskless asset. This paper aims to (i) maximize the total expected discounted utility of consumption of the investor in an infinite time horizon, (ii) determine the optimal net debt ratio for an investor under an economy that faces financial crisis, (iii) determine the optimal investment strategies of an investor who invest in an economy that is exposed to both diffusion and credit risks, and (iv) determine the real wealth of an investor. The optimal consumption and investment strategies as well as optimal net debt ratio under power utility function were obtained. We found that investment in FA can hedge the credit risks in the stock market investment portfolio. We also found that the investment portfolio in FA depends inversely on the optimal debt ratio of the economy and directly on the investment portfolio in FM. The numerical implementation of our models using real data from ten companies collected from Nigerian Stock Exchange is also presented in this paper.



## 1. Introduction

The global financial crisis of 2008 is already causing a considerable slowdown in the economies of most developed and developing countries of the World such as Nigeria. This paper focus on solving the problem of financial crisis facing the world economy. The financial crisis of 2008 has had adverse effects on the World economy and is still affecting most part of the World even today. The financial crisis on the world economy was not new. According to [1], there have been 124 economic crises between 1970 and 2007 that have occurred in the developing world. The crisis came on the back of the Clinton era deregulation of the financial markets which was later globalized. There was low interest rates in US between 2001 and 2007 to ease the US economy and boost consumer confidence. This was later used as an economic palliative measure to stimulate economic growth. Several non-credits worthy individuals and households took advantage of it to obtained mortgages which they do not have the ability to pay back. This led to a high credit risk. In order to solve this problem, this paper put into consideration backup security on loan borrowed by the investor.

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The Nigerian economy is facing a lot of challenges due to adverse effects of economic recession of 2008. In October, 2008, the then Central Bank of Nigeria governor, Professor Charles Soludo, said that the Nigerian economy will not suffer adverse effects of the crisis, since the economy has low integration with the global economy, see [2]. Soludo who pronounce all Nigerian banks and economy safe, do not take into account that the Nigerian economy was (and still is) over dependent on crude oil exports for foreign exchange earnings and revenue. This over dependence of crude oil made the economy vulnerable to shocks in oil prices. The Nigerian Capital Market was also in a bubble, the prices of stocks went down drastically and Nigerian banks became vulnerable to loan risks. Stock markets were sensitive to both national and international activities and react immediately to them. Between 2006 and 2016, the Nigeria stock prices declined with over 31.2%. The Nigerian stock market witnessed unprecedented growth in total market capitalization between 2004 and second quarter of 2008. But, in 2012 there was a total meltdown of the economy which affected the Nigerian stock market that experience serious downturn activities. Investors were pulling out their investment from the stock market which resulted to drop in the stock prices. Some of the main causes of economic recession include: high rate of inflation, accumulation of debts servicing especially foreign debts, high interest rate, fall in aggregate demand, fall in wages and income, and mass unemployment and general loss of confidence on the government due to economic indices.

The business activities of the financial institutions in Nigeria are exposed to several risks. These risks include: economic risk, political risk, policy risk, sectoral risk, interest rate risk, inflation risk, regulatory risk, operational risk, currency risk, environmental risk, risk create by Niger-Delta youths, corruption, etc. In this paper, we intend to classify these risks into four (i.e., inflation, investment, fixed assets and income growth risks) and address these problems. We assume that the economy has two distinct markets: financial and fixed asset markets. The FM has two underlying assets: a riskless asset and $n$ classes of stocks. The FA market involves $m$ number of FA which include lands and buildings in different locations in the various states in Nigeria. The prices of land and housing vary in one location and another and they are exposed to different risk levels.

One of the ways of reducing investment risks in an investment portfolio is through investment diversification. In this paper, we consider an investor's problem who chooses to invest his or her assets in financial market (FM) and FAs. The underlying assets in the FM are riskless (cash account) and multiple risky assets (stocks). One of the aims of the investor is to maximize the total expected discounted utility of consumption over an infinite time horizon under four control variables: debt ratio; investment strategies in stocks, investment strategies in FAs and consumption plan.

In this paper, the stock price process, the FA price and income growth rate of the investor are all stochastic with time-dependent drift terms and they follow a linear growth rate. The dynamics of investment in FA follows a geometric Brownian motion. The total asset value of the investor involves the addition of assets from the FM and FAs. The resulting nominal wealth is the difference between the total asset value and the liability. Since the economy is exposed to inflation, we consider the real wealth rather the nominal wealth. The dynamics of real net wealth process was solved using stochastic dynamic programming techniques of four control variables. The resulting Hamilton–Jacobi–Bellman (HJB) equation was derived with transversality condition. The optimal investment in FM and FA market, consumption and debt ratio were obtained by assuming that the investor is risk averse and chooses CRRA utility function. As a result, optimal investment in both FM and FA market, optimal debt ratio and optimal consumption were obtained. The explicit form of our HJB equation was also obtained. Data were collected from [1,3] and [4] to validate our models. In order to obtain the values of our parameters, the data were analyzed using Statistical Package for Social Sciences (SPSS). The values of the parameters obtained were used in our resulting models and the models were solved using MatLab 7.5.0.

But, the major challenges that economies are facing are how to manage inflation risks, income risks, investment risks, consumption process and incidence of credit risks in a financial crisis. This paper aim at tackling these problems.

We now give the highlights of our research work as follows:

1. debt ratio of an investor is considered,
2. backup security on liability is considered,
3. four background risks: inflation, investment, fixed assets and income growth risks are considered,
4. real wealth and consumption plan of an investor are considered,
5. optimal debt of an investor is obtained,
6. optimal investment and optimal consumption are obtained,
7. empirical data were used to analyze the resulting models,
8. theoretical and empirical studies of our models are carried out.

The remainder of the paper is organized as follows. In Section 2, we present the literature review. Our probability space, inflation dynamics, the financial models, income growth rate, dynamics of asset returns and the asset value are presented in Section 3. Section 4 presents the consumption plan, debt process and collateral security. Section 5 deals with the dynamics of the net wealth process of an investor. The optimal controls, value functions, optimal investments, optimal net debt ratio and optimal consumption are presented in Section 6. Section 7 presents optimal investment of an investor in FM and FA in terms of optimal debt ratio, and the explicit form of the HJB equation. Section 8 presents the empirical results of our models. Section 9 concludes the paper.

## 2. Literature review

This paper is centered on the following: inflation index, optimal investment, asset allocation, debt management and consumption plan. Hence, the literature review on our problem is structured in the following three subsections:

- first, is the inflation index,
- second, optimal investment and asset allocation,
- last, debt ratio and consumption plan of an investor.

### *2.1. Inflation index*

In an inflation environment, it is imperative for investors to know her real wealth rather than nominal wealth. Real wealth will guide investors in making the right decisions on his or her investment. [5] used a stochastic dynamic programming approach to model a DC pension fund in a complete financial market with stochastic investment opportunities and two background risks: income risk and inflation risk. He gave a closed form solution to the asset allocation problem and analyzed the behavior of the optimal portfolio with respect to income and inflation. [6] modeled inflation index that involves inflation uncertainty. They assumed that the inflation rate follows the Ornstein–Uhlenbeck process. [7] and [8] considered a one factor stochastic inflation dynamics. [9] considered a three-factor inflation index with jumps. In this work, we consider a four-factor stochastic inflation index for an investor.

### *2.2. Optimal investment and asset allocation*

Here, we give the literature review on optimal investment and asset allocation problem of an investor. [10] develops a simple framework for analyzing a finite-horizon investor's asset allocation problem under inflation environment and when only nominal assets are available. They found that the hedging demands depend on the investor's horizon and risk aversion and on the maturities of the bonds included in their portfolio. They further found that both the optimal stock-bond mix and the optimal bond maturity depend on the investor's horizon and risk aversion. [11] studied the implications of jumps in prices and volatility on investment strategies. They provide analytical solutions to the optimal portfolio problem. They found that the event risk practically affects the optimal strategy. [12] investigates optimal intertemporal asset allocation and location decisions for investors making taxable and tax-deferred investments. They showed that a strong preference for holding taxable bonds in the tax-deferred account and equity in the taxable account, reflects the higher tax burden on taxable bonds relative to equity. They further found that optimal portfolio decisions are a function of age and tax-deferred wealth. They also found that the proportion of total wealth allocated to equity is inversely related to the fraction of total wealth in tax-deferred accounts. [13] considered an optimal life-cycle asset allocation problem and showed that a life-cycle model with realistically calibrated uninsurable labor income risk and moderate risk aversion coefficient can simultaneously match stock market participation rates and asset allocation decisions conditional on participation. Their model studied Epstein–Zin preferences, a fixed stock market entry cost, and moderate heterogeneity in risk aversion. They found that households with low risk aversion smooth earnings shocks with a small buffer stock of assets, and consequently most of them never invest optimally in equities. They also found that the marginal stockholders are more risk averse, and as a result they do not invest their portfolios fully in stocks. [14] considered the formulation and studied a continuous time stochastic model of optimal asset allocation for a DC pension fund with a minimum guarantee. [15] considered computationally efficient approach to constrained discrete-time dynamic asset allocation over multiple periods. [16] considered the dynamic mean–variance portfolio problem and derived its time-consistent solution using dynamic programming techniques. They provide a fully analytical simple characterization of the dynamically optimal mean–variance portfolios within a general incomplete-market economy. They also identify a probability measure that incorporates intertemporal hedging demands and facilitates tractability. They found that a calibration exercise shows that the mean–variance hedging demands are economically significant. There are extensive literature that exist on the area of asset allocation problem. This can be found in [17–23].

### *2.3. Debt ratio and consumption plan of an investor*

In this subsection, we consider the literature review on debt and consumption plan. [24] studied the optimal debt ratio and consumption plan for an investor undergoing financial crisis. The impact of labor market condition was also studied. He assumed that the production rate function of the investor is stochastic and being influenced by the government policy, employment and unanticipated risks. The aim of the paper was to maximize the total expected discounted utility of consumption in the infinite time horizon by considering two control variables: debt ratio and consumption plans. [25] considered a stochastic optimal control model and optimal debt ratio management strategies of an investor in a financial crisis. They considered productivity of capital, asset return, interest rate and market regime switches of an investor. The utility of terminal wealth was optimized under debt ratio. [26] considered stochastic optimal control and dynamic programming technique. He derived an optimal debt of an investor under recession. He further stated that the deviation of the actual debt from the optimal, will serve as a warning signal for a financial crisis. [27] illustrated the importance of mathematical techniques of stochastic optimal control on solving problem on mortgage crisis. [28] developed and applied mathematical model to solving an economic problem in the presence of financial crisis.

## 3. The models

In this section, we present our probability space, dynamics of the underlying assets in FM, dynamics of FA prices, the asset value, the debt process and the collateral security of an investor.

### 3.1. Our probability space

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, $t \in [0, T]$, where $T < \infty$ is the terminal time of the operations and

$$\mathbf{W}(t) = (\mathbf{W}_I(t), \mathbf{W}_S(t), \mathbf{W}_P(t), \mathbf{W}_\beta(t))'$$

defined on a given filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}(\mathcal{F}), \mathbf{P})$, where $\mathcal{F}_t = \sigma(\tilde{\mathbf{W}}(s) : s \le t)$ and $\mathbf{F}(\mathcal{F}) = \{\mathcal{F}_t : t \in [0, T]\}$, is a $k+n+m+d$-dimensional Brownian motion with respect to inflation risks, stock market risks, FAs risks and income risks of an investor, respectively at time $t$ and $\mathbf{W}(t)$ is a $k+m+n+d$-dimensional Brownian motion with respect to inflation risks $\mathbf{W}_I(t)$, an $k$-dimensional Brownian motion, stock market sources of risks $\mathbf{W}_S(t)$, an $n$-dimensional Brownian motion, prices of FA sources of risks $\mathbf{W}_P(t)$, an $m$-dimensional Brownian motion and the income growth rate source of risk $\mathbf{W}_\beta(\mathrm{t})$ is the $d$-dimensional Brownian motions at time $t$. $\mathbf{P}$ denotes the real world probability measure, the sign "′", denotes transpose.

### 3.2. The inflation rate dynamics

In this subsection, we give the following dynamics of the inflation rate at time $t$, which we assumed to be correlated with inflation risks, stock market risks, FA market risks and income risks:

$$\begin{aligned} dI(t) &= I(t)(\mu_I(t)dt + \sigma_{1,I}(t)d\mathbf{W}_I(t) + \sigma_{2,I}(t)d\mathbf{W}_S(t) \\ &+ \sigma_{3,I}(t)d\mathbf{W}_P(t) + \sigma_{4,I}(t)d\mathbf{W}_\beta(t)), I(0) = I_0 > 0, \end{aligned} \tag{1}$$

where $\mu_I(t) = r(t) - \bar{r}(t) + \sigma_{1,I}(t)\theta_I(t) + \sigma_{2,I}(t)\theta_S(t) + \sigma_{3,I}(t)\theta_P(t) + \sigma_{4,I}(t)\theta_\beta(t)$ is the expected inflation index at time $t$, $r(t)$ is the nominal interest rate at time $t$ and $\bar{r}(t)$ is the real interest rate at time $t$, $\sigma_{1,I}(t) \in [\mathbb{R}^k \times [0, T]]$ is the volatility of price index with respect to inflation source risks, $\mathbf{W}_I(t)$, $\sigma_{2,I}(t) \in [\mathbb{R}^n \times [0, T]]$ is the volatility of price index with respect to source of stock market risks, $\mathbf{W}_S(t)$ at time $t$, $\sigma_{3,I}(t) \in [\mathbb{R}^m \times [0, T]]$ is the volatility of price index with respect to source of fixed asset risks, $\mathbf{W}_P(t)$ at time $t$, $\sigma_{4,I}(t) \in [\mathbb{R}^d \times [0, T]]$ is the volatility of price index with respect to source of income growth risks, $\mathbf{W}_\beta(t)$ at time $t$, $\theta_I(t) = (\theta_{I,1}(t), \theta_{I,2}(t), \ldots, \theta_{I,k}(t))'$ is the market price of inflation risks at time $t$, $\theta_S(t) = (\theta_{S,1}(t), \theta_{S,2}(t), \ldots, \theta_{S,n}(t))'$ is the market price of stock risks at time $t$, $\theta_P(t) = (\theta_{P,1}(t), \theta_{P,2}(t), \ldots, \theta_{P,m}(t))'$ is the market price of fixed asset risks at time $t$, $\theta_\beta(t) = (\theta_{\beta,1}(t), \theta_{\beta,2}(t), \ldots, \theta_{\beta,d}(t))'$ is the market price of income growth risks at time $t$,

$$\theta(t) = \begin{pmatrix} \theta_I(t) \\ \theta_S(t) \\ \theta_P(t) \\ \theta_\beta(t) \end{pmatrix} \tag{2}$$

is the market price of risks and $\sigma_Z(t) = (\sigma_{1,I}(t), \sigma_{2,I}(t), \sigma_{3,I}(t), \sigma_{4,I}(t))$ is the volatility vector of the price index, such that $\mu_I(t) = r(t) - \bar{r}(t) + \sigma_Z(t)\theta(t)$. Here, we assume that the consumer price index may be affected by the fluctuation of inflation, stock prices, prices of fixed assets and income growth of an investor. We now express (1) compactly as follows:

$$dI(t) = I(t)\left(\mu_I(t)dt + \sigma_Z(t)d\mathbf{W}(t)\right), I(0) = I_0 > 0. \tag{3}$$

### 3.3. Dynamics of the underlying assets in FM

Consider a problem of maximizing the expected utility of consumption in a infinite-time horizon by investing an amount $H_S(t)$ in a market that is made up of a set of risky assets and a riskless asset. In other words, the investor selects an amount to be invested in $n$ risky assets with price process $S(t) = [S_1(t), \ldots, S_n(t)]'$ which is correlated with inflation, stock market, fixed assets and income growth risks, and a riskless asset with price process $B(t)$ at time $t \in [0, \infty)$. The consumption path of an investor is also considered.

The dynamics of the underlying assets in the FM are given by the following:

$$dB(t) = r(t)B(t)dt, B(0) = 1, \tag{4}$$

for the riskless asset and

$$\begin{aligned} dS(t) &= S(t)(\mu(t)dt + \sigma_1(t)d\mathbf{W}_I(t) + \sigma_2(t)d\mathbf{W}_S(t) + \sigma_3(t)d\mathbf{W}_P(t) \\ &+ \sigma_4(t)d\mathbf{W}_\beta(t)), S(0) = s_0, \end{aligned} \tag{5}$$

for stocks, with a linear growth rate of interest $r(t) = r_0 + r_1 t$, where $r_0$ is the initial rate of interest and $r_1$ is the growth rate of interest over time, the expected growth rate of stocks is $\mu_i(t) = \mu_{i0} + \mu_{i1}t$, $i = 1, \ldots, n$, where $\mu_{i0}$ is the initial rate of stock $i$ and $\mu_{i1}$ is the growth rate of stock $i$ over time. We can express it in vector form as follows: $\mu(t) = [\mu_1(t), \dots, \mu_n(t)]'$ and volatilities of stocks with respect to inflation risks $\sigma_1(t) \in [\mathcal{R}^{n\times k} \times [0,T]]$, volatilities of stocks with respect to stock market risks $\sigma_2(t) \in [\mathcal{R}^{n\times n} \times [0,T]]$, volatilities of stocks with respect to fixed assets risks $\sigma_3(t) \in [\mathcal{R}^{n\times m} \times [0,T]]$, volatilities of stocks with respect to income growth risks $\sigma_4(t) \in [\mathcal{R}^{n\times d} \times [0,T]]$.

We now re-write (5) compactly as follows:

$$dS(t) = S(t)\left(\mu(t)dt + \sigma(t)d\mathbf{W}(t)\right), S(0) = s_0, \tag{6}$$

where $\sigma(t) = (\sigma_1(t), \sigma_2(t), \sigma_3(t), \sigma_4(t))$, $\Sigma(t) = \sigma(t)\sigma(t)'$.

We assume that $\Sigma(t)$ is a nonsingular and positive definite matrix.

### 3.4. The dynamics of the price of FA

In this subsection, we consider the FA and its price process at time $t$. Let $P(t)$ be the price process of FA at time $t$. We assume that the FA price $P(t)$ satisfies the dynamics

$$dP(t) = P(t)\left(\alpha(t)dt + \sigma_P(t)d\mathbf{W}(t)\right), P(0) = p_0 \in \mathcal{R}_+, \tag{7}$$

where $P(t) = [P_1(t), \dots, P_m(t)]$, $\alpha(t) = \alpha_0 + \alpha_1 t$ is the return rate of the FA at time $t$, $\alpha_0 = [\alpha_{10}, \alpha_{20}, \dots, \alpha_{m0}]'$ is the initial growth rate vector of FA and $\alpha_1 = [\alpha_{11}, \alpha_{21}, \dots, \alpha_{m1}]'$ is the growth rate vector over time of FA due to some macro- and micro-economic factors such as inflation, government policies, natural effects, economic growths, e.t.c, $\sigma_P(t) = (\sigma_P^1(t), \sigma_P^2(t), \sigma_P^3(t), \sigma_P^4(t))$ is the volatility tensor of the FA, where $\sigma_P^1(t) \in [\mathcal{R}^{m\times k} \times [0,T]]$, $\sigma_P^2(t) \in [\mathcal{R}^{m\times n} \times [0,T]]$, $\sigma_P^3(t) \in [\mathcal{R}^{m\times m} \times [0,T]]$, $\sigma_P^4(t) \in [\mathcal{R}^{m\times d} \times [0,T]]$ and $\Sigma_P(t) = \sigma_P(t)\sigma_P(t)'$ is a positive definite and nonsingular matrix.

### 3.5. The income growth rate of an investor

Here, we assume that the income growth rate is a stochastic process and affected by economic, scientific, human, environmental and political factors. The rate of unemployment was identified by [24] as a major factor that influence the income rate. He stated that when the unemployment rate $\omega$ is low, the economy can be seen to be expanding, and production rate will be higher than expected, while increase in unemployment rate $\omega$ can bring about recession, and the production rate will be lower than expected. It is assumed that the income growth rate $\beta(t)$ follows the dynamics

$$\begin{aligned} &d\beta(t) = [f(\beta(t)) + \beta(t)\eta(\omega)]dt + \sigma_\beta(t)d\mathbf{W}(t), \\ &\beta(0) = \beta_0, \end{aligned} \tag{8}$$

where $f(\beta(t)) : \mathcal{R} \longrightarrow \mathcal{R}$ is the expected production rate and satisfies the following ordinary differential equation (ODE):

$$df(\beta(t)) = \epsilon f(\beta(t))dt, f(\beta(0)) = 1, \tag{9}$$

$\epsilon$ is a constant, $\eta(\omega) : \mathcal{R} \longrightarrow \mathcal{R}$ is a continuous function in $\omega$ and represents the effect of unemployment rate to production process of an investor. According to [24], if $\eta(\omega) > 0$, it implies that the investment environment of the investor is expanding, if $\eta(\omega) < 0$, it implies that the investment environment of the investor in recession and if $\eta(\omega) = 0$, it implies that the investment environment of the investor is in critical position. The tensor $\sigma_\beta(t) = [\sigma_\beta^1(t), \sigma_\beta^2(t), \sigma_\beta^3(t), \sigma_\beta^4(t)]$ is the volatility of the income growth rate, where $\sigma_\beta^1(t) \in [\mathcal{R}^k \times [0,T]]$, $\sigma_\beta^2(t) \in [\mathcal{R}^n \times [0,T]]$, $\sigma_\beta^3(t) \in [\mathcal{R}^m \times [0,T]]$ and $\sigma_\beta^4(t) \in [\mathcal{R}^d \times [0,T]]$.

### 3.6. The asset value

**Definition 1.** Let $A(t)$ be the total asset values of an investor at time $t$, $H_F(t)$ be the asset in FA market at time $t$. Then, the total asset value is defined as

$$A(t) = H_S(t) + H_F(t). \tag{10}$$

Since $A(t)$ is the total asset invested by an investor in FM and FA at time $t$, let $\Delta_S(t) = [\Delta_{S1}(t), \dots, \Delta_{Sn}(t)]$ be the fund invested in stock, $S(t)$ at time $t$, $\Delta_F(t) = [\Delta_{F1}(t), \dots, \Delta_{Fm}(t)]$ the fund invested in FA at time $t$ and the remainder $\Delta_0(t) = A(t) - \Delta_F(t)\mathbf{e}_m - \Delta_S(t)\mathbf{e}$ is invested in riskless asset at time $t$, where $\mathbf{e} = [1, \dots, 1]' \in \mathcal{R}^n$ and $\mathbf{e}_m = [1, \dots, 1]' \in \mathcal{R}^m$. From now on, we assume that $k = n = m = d$. We now have the following definitions.

**Definition 2.** The wealth dynamics of assets generated from the FM is defined as follows:

$$dH_S(t) = \Delta_0(t)\frac{dB(t)}{B(t)} + \Delta_S(t)\frac{dS(t)}{S(t)}, H_S(0) = H_{S0} \in \mathcal{R}_+. \tag{11}$$

Using (4) and (6) on (11), we obtain the following:

$$\begin{aligned} dH_S(t) = &(r(t)A(t) - \Delta_F(t)r(t)\mathbf{e}_m + \Delta_S(t)(\mu(t) - r(t)\mathbf{e}))dt \\ &+ \Delta_S(t)\sigma(t)d\mathbf{W}(t), H_S(0) = H_{S0} \in \mathcal{R}_+. \end{aligned} \tag{12}$$

**Definition 3.** The wealth dynamics of assets generated from the FA market is defined as follows:

$$dH_F(t) = H_F(t)\Delta_F(t)\frac{dP(t)}{P(t)}, H_F(0) = H_{F0} \in \mathcal{R}_+. \tag{13}$$

Using (7) on (13), we obtain the following:

$$\begin{aligned} &dH_F(t) = \Delta_F(t)H_F(t)\alpha(t)dt + \Delta_F(t)H_F(t)\sigma_P(t)d\mathbf{W}(t), \\ &H_F(0) = H_{F0} \in \mathcal{R}_+. \end{aligned} \tag{14}$$

**Proposition 1.** *The rate of change of the asset values of an investor is*

$$\begin{aligned} &dA(t) = [r(t)A(t) + \Delta_S(t)(\mu(t) - r(t)\mathbf{e}) + (A(t) - H_S(t))\Delta_F(t)(\alpha(t) - r(t)\mathbf{e}_m)]dt \\ &+ [\Delta_S(t)\sigma(t) + (A(t) - H_S(t))\Delta_F(t)\sigma_P(t)]d\mathbf{W}(t). \end{aligned} \tag{15}$$

**Proof.** Using Definition 1, we have the following by taking the differential of both sides of (10):

$$dA(t) = dH_S(t) + dH_F(t). \tag{16}$$

By substituting (12) and (14) into (16), it therefore follows that

$$\begin{aligned} &dA(t) = [r(t)A(t) + \Delta_S(t)(\mu(t) - r(t)\mathbf{e}) + H_F(t)\Delta_F(t)(\alpha(t) - r(t)\mathbf{e}_m)]dt \\ &+ [\Delta_S(t)\sigma(t) + H_F(t)\Delta_F(t)\sigma_P(t)]d\mathbf{W}(t). \end{aligned} \tag{17}$$

Using the fact that $H_F(t) = A(t) - H_S(t)$, then (17) becomes

$$\begin{aligned} &dA(t) = [r(t)A(t) + \Delta_S(t)(\mu(t) - r(t)\mathbf{e}) + (A(t) - H_S(t))\Delta_F(t)(\alpha(t) \\ &- r(t)\mathbf{e}_m)]dt + [\Delta_S(t)\sigma(t) + (A(t) - H_S(t))\Delta_F(t)\sigma_P(t)]d\mathbf{W}(t). \quad \square \end{aligned} \tag{18}$$

(18) is the dynamics of the total asset value of an investor expressed in terms of $H_S(t)$.

## 4. The consumption plan, collateral security and debt process

In this section, we consider the consumption plan, the debt process and collateral security of an investor.

### 4.1. The consumption plan and income of an investor

In this subsection, we consider the consumption plan of an investor. If the investor chooses to consume continuously at the rate $c(t)$ at time $t$, then the dynamics is given by

$$dC(t) := c(t)X(t)dt, \tag{19}$$

where $X(t)$ is the net wealth process of an investor at time $t$ and $C(t)$ the amount consumed by an investor at time $t$.

Let $E(t)$ be the income process of an investor with a growth rate $\beta(t)$. Then, the change in $E(t)$ with respect to $t$ is given by the product of the income growth rate and asset value. That is, the change in income process is given by

$$dE(t) = \beta(t)A(t)dt, \tag{20}$$

where $\beta(t)$ satisfies (8).

### 4.2. The collateral security

Here, we consider the collateral security of an investor on her liability. If the investor chooses to lend an amount of money $L(t)$ with interest rate $r_D(t)$ at time $t$, the amount borrowed is assumed to be back up with a collateral security. We now have the following definition:

**Definition 4.** Let $K(t)$ be the value of the collateral security on the debt $L(t)$ at time $t$, then we define $K(t)$ as

$$dK(t) = \xi(t)K(t)dt, K(0) = K_0 > 0$$

where $\xi(t)$ is the growth rate of the collateral security at time $t$.

### 4.3. The debt process of an investor

In this subsection, we consider the debt process of an investor over time. Hence, we define the debt dynamics of an investor using (19) and (20) as follows:

$$dL(t) = r_D(t)L(t)dt + dC(t) - dE(t). \tag{21}$$

**Definition 5.** The net debt of an investor $Z(t)$ at time $t$ is defined as

$$Z(t) = L(t) - K(t).$$

**Proposition 2.** *The dynamics of the net debt of an investor is*

$$dZ(t) = r_D(t)Z(t)dt + (r_D(t) - \xi(t))K(t)dt + c(t)X(t)dt - \beta(t)A(t)dt.$$

**Proof.** We commence by expressing (21) in terms of $K(t)$. Since the amount, $K(t)$ is to be deducted from the loan $L(t)$, it then follows that

$$d(L(t) - K(t)) = r_D(t)L(t)dt + dC(t) - dE(t) - dK(t). \tag{22}$$

Using Definitions 4, 5 and (20), we have

$$dZ(t) = r_D(t)Z(t)dt + (r_D(t) - \xi(t))K(t)dt + c(t)X(t)dt - \beta(t)A(t)dt. \quad \square \tag{23}$$

(23) is the dynamics of our net debt at time $t$. One of the aims of this paper is to optimize the net debt $Z(t)$ rather than $L(t)$, since the later has to a great extent protected by $K(t)$.

## 5. The wealth process of an investor

In this section, we consider the dynamics of the net nominal and real wealth process of the investor at time $t$.

### 5.1. The dynamics of the net nominal wealth process

In this subsection, we present the net wealth process of the investor.

**Definition 6.** Let $X(t)$ be the net wealth process of an investor at time $t$ defined as the difference between the asset values $A(t)$ and debt $L(t)$ at time $t$. Mathematically,

$$X(t) = A(t) - L(t).$$

**Proposition 3.** *Suppose that $X(t)$ is the net wealth process of an investor, then*

$$\begin{aligned}
\frac{dX(t)}{X(t)} &= [r(t) + z(t)(r(t) - r_D(t)) + (\xi(t) + r(t) - r_D(t))g_K(t) \\
&+ \pi_S(t)(\mu(t) - r(t)\mathbf{e}) + (1 + z(t) + g_K(t) - g_S(t))\pi_F(t)(\alpha(t) \\
&- r(t)\mathbf{e}_m) + \beta(t)(1 + z(t) + g_K(t)) - c(t)]dt + [\pi_S(t)\sigma(t) \\
&+ (1 + z(t) + g_K(t) - g_S(t))\pi_F(t)\sigma_P(t)]d\mathbf{W}(t), \\
X(0) &= x_0 > 0.
\end{aligned} \tag{24}$$

*where*

$\pi_S(t) = \frac{\Delta_S(t)}{X(t)}$ *is the proportion of wealth invested in risky assets at time* $t$,
$\pi_F(t) = \frac{\Delta_F(t)}{X(t)}$ *is the proportion of wealth invested in FA assets at time* $t$,
$z(t) = \frac{Z(t)}{X(t)}$ *is the net debt ratio at time* $t$,
$g_K(t) = \frac{K(t)}{X(t)}$ *is the collateral security ratio at time* $t$,
$g_S(t) = \frac{H_S(t)}{X(t)}$ *is the fraction of wealth invested in the FM at time* $t$.

**Proof.** By using Definition 6, we have that

$$X(t) = A(t) - L(t) = H_S(t) + H_F(t) - L(t). \tag{25}$$

Taking the differential of both sides of (25), we obtain the dynamics of the net wealth process of an investor as follows:

$$dX(t) = dA(t) - dL(t) = dA(t) - d(Z(t) + K(t)). \tag{26}$$

Substituting in (18) and (23) into (26), we have the following:

$$\begin{aligned} dX(t) &= [r(t)A(t) - r_D(t)Z(t) - r_D(t)K(t) + \xi(t)K(t) + \Delta_S(t)(\mu(t) - r(t)\mathbf{e}) \\ &+ (A(t) - H_S(t))\Delta_F(t)(\alpha(t) - r(t)\mathbf{e}_m) + \beta(t)A(t) - C(t)]dt \\ &+ [\Delta_S(t)\sigma(t) + (A(t) - H_S(t))\Delta_F(t)\sigma_P(t)]d\mathbf{W}(t), X(0) = x_0 > 0. \end{aligned} \tag{27}$$

Observe that $A(t) = X(t) + Z(t) + K(t)$. Then, substituting it into (27), we have

$$\begin{aligned} dX(t) &= [r(t)X(t) + Z(t)(r(t) - r_D(t)) + (\xi(t) + r(t) - r_D(t))K(t) \\ &+ \Delta_S(t)(\mu(t) - r(t)\mathbf{e}) + (X(t) + Z(t) + K(t) \\ &- H_S(t))\Delta_F(t)(\alpha(t) - r(t)\mathbf{e}_m) + \beta(t)(X(t) + Z(t) + K(t)) - C(t)]dt \\ &+ [\Delta_S(t)\sigma(t) + (X(t) + Z(t) + K(t) - H_S(t))\Delta_F(t)\sigma_P(t)]d\mathbf{W}(t), \\ X(0) &= x_0 > 0. \end{aligned} \tag{28}$$

It then follows that

$$\begin{aligned} \frac{dX(t)}{X(t)} &= [r(t) + z(t)(r(t) - r_D(t)) + (\xi(t) + r(t) - r_D(t))g_K(t) \\ &+ \pi_S(t)(\mu(t) - r(t)\mathbf{e}) + (1 + z(t) + g_K(t) - g_S(t))\pi_F(t)(\alpha(t) \\ &- r(t)\mathbf{e}_m) + \beta(t)(1 + z(t) + g_K(t)) - c(t)]dt + [\pi_S(t)\sigma(t) \\ &+ (1 + z(t) + g_K(t) - g_S(t))\pi_F(t)\sigma_P(t)]d\mathbf{W}(t), \\ X(0) &= x_0 > 0. \quad \square \end{aligned} \tag{29}$$

**Remark 1.** Note that $K(t)$ is the using asset while $H_S(t)$ and $H_F(t)$ are holding securities.

### *5.2. The real wealth dynamics of an investor*

In this subsection, we consider the real wealth of the investor. An investor is assumed to be interested in the utility of her real wealth level since the buying power of the nominal wealth is diminished by the inflation, see [9].

**Definition 7.** The real wealth of the investor, $\bar{X}(t)$ at time $t$ is defined as the ratio of the nominal wealth to the price index. Mathematically,

$$\bar{X}(t) = \frac{X(t)}{I(t)}. \tag{30}$$

**Proposition 4.** *The dynamics of the real wealth of the investor at time $t$ is*

$$\begin{aligned} d\bar{X}(t) &= d\left(\frac{X(t)}{I(t)}\right) = \bar{X}(t)[r(t) + z(t)(r(t) - r_D(t)) + (\xi(t) + r(t) \\ &- r_D(t))g_K(t) + \pi_S(t)(\mu(t) - r(t)\mathbf{e}) + (1 + z(t) + g_K(t) \\ &- g_S(t))\pi_F(t)(\alpha(t) - r(t)\mathbf{e}_m) + \beta(1 + z(t) + g_K(t)) - c(t) \\ &- \mu_I(t) + \sigma_Z(t)\sigma_Z(t)' - (\pi_S(t)\sigma(t) + (1 + z(t) + g_K(t) - g_S(t))\pi_F(t)\sigma_P(t))\sigma_Z(t)']dt \\ &+ \bar{X}(t)(\pi_S(t)\sigma(t) + (1 + z(t) + g_K(t) - g_S(t))\pi_F(t)\sigma_P(t) - \sigma_Z(t))d\mathbf{W}(t), \\ \bar{X}(0) &= \bar{x}_0 \in \mathbb{R}_+. \end{aligned} \tag{31}$$

**Proof.** Using Definition 7 and finding the differential of both sides of (30), we have

$$d\bar{X}(t) = d\left(\frac{X(t)}{I(t)}\right). \tag{32}$$

By the application of Itô formula for quotient rule on (32) and using (3) and (44), the result follows immediately. $\square$

For the sake of simplicity, from now on we will not indicate the functional dependences, unless it is necessary to do so. It now follows that (32) will become:

$$\begin{aligned} d\bar{X} &= d\left(\frac{X}{I}\right) = \bar{X}[r + z(r - r_D) + (\xi + r - r_D)g_K + \pi_S(\mu - r\mathbf{e}) \\ &+ (1 + z + g_K - g_S)\pi_F(\alpha - r\mathbf{e}_m) + \beta(1 + z + g_K) - c - \mu_I + \sigma_Z\sigma_Z' \\ &- (\pi_S\sigma + (1 + z + g_K - g_S)\pi_F\sigma_P)\sigma_Z']dt \\ &+ \bar{X}(\pi_S\sigma + (1 + z + g_K - g_S)\pi_F\sigma_P - \sigma_Z)d\mathbf{W}(t), \bar{X}(0) = \bar{x}_0 \in \mathbb{R}_+. \end{aligned} \tag{33}$$

## 6. The optimal control problems

We now consider the admissible strategies, optimal controls and value function of our problem.

### 6.1. The admissible strategies

For admissible net debt ratio strategy $z(t)$, we assume that for all $T \in (0, \infty)$,

$$E \int_0^T z(t)^2 dt < \infty. \tag{34}$$

For admissible portfolio strategy in FM $\pi_S(t)$ and for all $T \in (0, \infty)$, we have

$$E \int_0^T \pi_S(t)\pi_S(t)' dt < \infty. \tag{35}$$

For admissible portfolio strategy in FA $\pi_F(t)$ and for all $T \in (0, \infty)$, we have

$$E \int_0^T \pi_F(t)\pi_F(t)' dt < \infty. \tag{36}$$

For the rate of consumption $c(t)$, we assume that is a non-negative and bounded above with upper bound being $\Theta$ such that $0 \le c(t) \le \Theta < \infty$.

A strategy $u(\cdot) = \{(z(t), \pi_S(t), \pi_F(t), c(t)) : t \ge 0\}$ which is progressively measurable with respect to $\{\mathbf{W}(t) : 0 \le s \le t\}$ is referred to as admissible strategy. We denote the collection of all admissible strategies by $\mathcal{A}$. It then follows that the set of all admissible strategies $\mathcal{A}$ can be defined as follows:

$$\begin{aligned} \mathcal{A} = \{u(t) = (z(t), \pi_S(t), \pi_F(t), c(t)) \in \mathcal{R} \times \mathcal{R}^n \times \mathcal{R}^m \times \mathcal{R} : E \int_0^T z(t)^2 dt < \infty; \\ E \int_0^T \pi_S(t)\pi_S(t)' dt < \infty; E \int_0^T \pi_F(t)\pi_F(t)' dt < \infty; 0 \le c(t) \le \Theta < \infty\}. \end{aligned} \tag{37}$$

### 6.2. The optimal controls and value function

In this subsection, we consider the optimal controls and the value function. We now set $\bar{X}(t) = x$ and $\beta(t) = \beta$. The value function is given by

$$G(x, \beta) := \sup_{u \in \mathcal{A}} F(x, \beta; u), \tag{38}$$

where

$$F(x, \beta; u) = E_{x,\beta} \int_0^\infty e^{-\delta t} V(c(t)x(t)) dt, \tag{39}$$

$\delta > 0$ is the discounted rate and $V(c(t)x(t))$ utility of consumption rate.

The investor's problem at time $t$ is to select the net debt rate, portfolio weights and consumption rate processes $\{z(s), \pi_S(s), \pi_F(s), c(s)\}_{t \le s \le \infty}$ that maximize the infinite horizon, expected utility of consumption, that is,

$$\begin{aligned} &G(\bar{X}(t), \beta(t)) = \max_{\{z(s), \pi_S(s), \pi_F(s), c(s) : t \le s \le \infty\}} E_t \left[ \int_t^\infty e^{-\delta s} V(c(s)x(s)) ds \right], \\ &\text{subject to} \\ &d \begin{bmatrix} \beta \\ \bar{X} \end{bmatrix} = \bar{m} dt + \bar{M} d\mathbf{W}(t), \bar{X}(0) = \bar{x}_0, \beta(0) = \beta_0, \end{aligned} \tag{40}$$

where

$$\bar{m} = \begin{bmatrix} f(\beta(t)) + \beta(t)\eta(\omega) \\ \bar{X}[r + z(r - r_D) + (\xi + r - r_D)g_K + \pi_S(\mu - r\mathbf{e}) \\ + (1 + z + g_K - g_S)\pi_F(\alpha - r\mathbf{e}_m) + \beta(1 + z + g_K) - c - \mu_I \\ + \sigma_Z\sigma_Z' - (\pi_S\sigma + (1 + z + g_K - g_S)\pi_F\sigma_P)\sigma_Z'] \end{bmatrix},$$

$$\bar{M} = \begin{bmatrix} \sigma_\beta \\ \bar{X}(\pi_S\sigma + (1 + z + g_K - g_S)\pi_F\sigma_P - \sigma_Z) \end{bmatrix}.$$

We now consider an investor that chooses power utility function. By applying stochastic dynamic programming approach and Itó Lemma for semimartingale processes, our Hamilton–Jacobi–Bellman equation characterized by the optimal solutions to the problem of an investor becomes

$$\begin{aligned}
&\mathcal{L}^u(G(x,\beta)) = \frac{1}{2}x^2(\pi_S\sigma + (1+z+g_K-g_S)\pi_F\sigma_P - \sigma_Z)(\pi_S\sigma \\
&+(1+z+g_K-g_S)\pi_F\sigma_P - \sigma_Z)'G_{xx} + \frac{1}{2}\sigma_\beta\sigma_\beta' G_{\beta\beta} + x(\pi_S\sigma \\
&+(1+z+g_K-g_S)\pi_F\sigma_P - \sigma_Z)\sigma_\beta' G_{x\beta} + x[r + z(r-r_D) + (\xi + r - r_D)g_K \\
&+\pi_S(\mu - r\mathbf{e}) + (1+z+g_K-g_S)\pi_F(\alpha - r\mathbf{e}_m) + \beta(1+z+g_K) - c - \mu_I \\
&+\sigma_Z\sigma_Z' - (\pi_S\sigma + (1+z+g_K-g_S)\pi_F\sigma_P)\sigma_Z']G_x + [f(\beta) + \beta\eta(\omega)]G_\beta,
\end{aligned} \tag{41}$$

with the transversality condition $\lim_{t\to\infty} E[G(t,\bar{X}(t),\beta(t))] = 0$, where $G_x = \frac{\partial G}{\partial x}, G_\beta = \frac{\partial G}{\partial \beta}, G_{xx} = \frac{\partial^2 G}{\partial x^2}, G_{x\beta} = \frac{\partial^2 G}{\partial x \partial \beta}, G_{\beta\beta} = \frac{\partial^2 G}{\partial \beta^2}$. The standard time-homogeneity argument for infinite-horizon problems gives

$$\begin{aligned}
G(\bar{X}(t),\beta(t)) &= \max_{\{z(s),\pi_S(s),\pi_F(s),c(s):t\le s\le\infty\}} E_t\left[\int_t^\infty e^{-\delta(s-t)}V(c(s)x(s))ds\right] \\
&= \max_{\{z(t-u),\pi_S(t-u),\pi_F(t-u),c(t-u):t\le s\le\infty\}} E_t\left[\int_0^\infty e^{-\delta u}V(C(t+u))du\right] \\
&= \max_{\{z(u),\pi_S(u),\pi_F(u),c(u):0\le u\le\infty\}} E_0\left[\int_0^\infty e^{-\delta u}V(C(u))du\right] \\
&\equiv U(\bar{X}(t),\beta(t)),
\end{aligned}$$

and is independent of time $t$. The third equality in the above argument makes use of the fact that the optimal control is Markovian. It then follows that $G(\bar{X}(t),\beta(t)) = e^{-\delta t}U(\bar{X}(t),\beta(t))$ and (41) reduces to the following equation for the time-homogeneous value function $U$:

$$\begin{aligned}
&\mathcal{L}^u(U(x,\beta)) = \frac{1}{2}x^2\pi_S\sigma(\pi_S\sigma)'U_{xx} + x^2(1+z+g_K \\
&-g_S)\pi_F\sigma_P(\pi_S\sigma)'U_{xx} + \frac{1}{2}x^2(1+z+g_K-g_S)^2\pi_F\sigma_P(\pi_F\sigma_P)'U_{xx} - x^2\pi_S\sigma\sigma_Z' U_{xx} \\
&-x^2(1+z+g_K-g_S)\pi_F\sigma_P\sigma_Z' U_{xx} + \frac{1}{2}\sigma_Z\sigma_Z' U_{xx} + \frac{1}{2}\sigma_\beta\sigma_\beta' U_{\beta\beta} + x(\pi_S\sigma \\
&+(1+z+g_K-g_S)\pi_F\sigma_P - \sigma_Z)\sigma_\beta' U_{x\beta} + x[r + z(r-r_D) + (\xi + r - r_D)g_K \\
&+\pi_S(\mu - r\mathbf{e}) + (1+z+g_K-g_S)\pi_F(\alpha - r\mathbf{e}_m) + \beta(1+z+g_K) - c - \mu_I \\
&+\sigma_Z\sigma_Z' - (\pi_S\sigma + (1+z+g_K-g_S)\pi_F\sigma_P)\sigma_Z']U_x + [f(\beta) + \beta\eta(\omega)]U_\beta,
\end{aligned} \tag{42}$$

with the transversality condition $\lim_{t\to\infty} E[e^{-\delta t}U(\bar{X}(t),\beta(t))] = 0$.

We now state formally that the value function satisfies the HJB equation (42) as follows:

$$\max_u\{\mathcal{L}^u U(x,\beta) - \delta U(x,\beta) + V(cx)\} = 0. \tag{43}$$

Now, for an arbitrary admissible strategy $u = (z,\pi_S,\pi_F,c)$, the objective function $F(\cdot)$ satisfies (39).

The maximization problem in (43) separates into one for net debt ratio $z(t)$, with first-order condition

$$\begin{aligned}
&x^2 z\pi_F\sigma_P(\pi_S\sigma)'U_{xx} + \frac{1}{2}x^2(2z + 2zg_K - 2zg_S + z^2)\pi_F\sigma_P(\pi_F\sigma_P)'U_{xx} \\
&-x^2 z\pi_F\sigma_P\sigma_Z' U_{xx} + xz\pi_F\sigma_P\sigma_\beta' U_{x\beta} + x[z\pi_F(\alpha - r\mathbf{e}_m) + (\beta + r - r_D)z - z\pi_F\sigma_P\sigma_Z']U_x
\end{aligned} \tag{44}$$

one for consumption rate $c(t)$:

$$\frac{\partial V(cx)}{\partial c} = xU_x \tag{45}$$

one for investment strategies $\pi_S(t)$:

$$\begin{aligned}
&\frac{1}{2}x^2\pi_S\sigma(\pi_S\sigma)'U_{xx} + x^2(1+z+g_K-g_S)\pi_F\sigma_P(\pi_S\sigma)'U_{xx} \\
&-x^2\pi_S\sigma\sigma_Z' U_{xx} + x\pi_S\sigma\sigma_\beta' U_{x\beta} + x[\pi_S(\mu - r\mathbf{e}) - \pi_S\sigma\sigma_Z']U_x
\end{aligned} \tag{46}$$

and one for investment strategies $\pi_F(t)$:

$$\begin{aligned}
&x^2(1+z+g_K-g_S)\pi_F\sigma_P(\pi_S\sigma)'U_{xx} + \frac{1}{2}x^2(1+z+g_K-g_S)^2\pi_F\sigma_P(\pi_F\sigma_P)'U_{xx} \\
&-x^2(1+z+g_K-g_S)\pi_F\sigma_P\sigma_Z' U_{xx} + x(1+z+g_K-g_S)\pi_F\sigma_P\sigma_\beta' U_{x\beta} \\
&+x[(1+z+g_K-g_S)\pi_F(\alpha - r\mathbf{e}_m) - (1+z+g_K-g_S)\pi_F\sigma_P\sigma_Z']U_x.
\end{aligned} \tag{47}$$

In order to determine the optimal debt ratio, optimal consumption plan, optimal portfolio weights, wealth and value function, we have to be more specific about the utility function $V$.

*6.3. Power utility*

Consider an investor with the following power utility, $V(C) = \frac{C^{1-\gamma}}{1-\gamma}$ for $C > 0$ and $V(C) = -\infty$ for $C < 0$ with CRRA coefficient $\gamma \in (0,1) \cup (1,\infty)$.

We now assume a solution to (43) to be of the form

$$U(x,\beta) = \frac{x^{1-\gamma}e^{h(\beta)}}{1-\gamma}, \tag{48}$$

so that

$$\begin{gathered}\frac{\partial U(x,\beta)}{\partial x} = (1-\gamma)U(x,\beta)/x, \frac{\partial^2 U(x,\beta)}{\partial x^2} = -\gamma(1-\gamma)U(x,\beta)/x^2,\\ \frac{\partial U(x,\beta)}{\partial \beta} = h_\beta U(x,\beta), \frac{\partial U(x,\beta)}{\partial x\beta} = h_\beta(1-\gamma)U(x,\beta)/x,\\ \frac{\partial^2 U(x,\beta)}{\partial \beta^2} = [(h_\beta)^2 + h_{\beta\beta}]U(x,\beta).\end{gathered} \tag{49}$$

Considering the four control variables $z, \pi_S, \pi_F$ and $c$, (43) turns out to be

$$\begin{aligned}0 = &\max_{z,\pi_S,\pi_F,c}\{-\delta U + \frac{1}{2}x^2\pi_S\sigma(\pi_S\sigma)'U_{xx} + x^2(1+z+g_K-g_S)\pi_F\sigma_P(\pi_S\sigma)'U_{xx}\\ &+\frac{1}{2}x^2(1+z+g_K-g_S)^2\pi_F\sigma_P(\pi_F\sigma_P)'U_{xx} - x^2\pi_S\sigma\sigma_Z'U_{xx} - x^2(1+z\\ &+g_K-g_S)\pi_F\sigma_P\sigma_Z'U_{xx} + x(\pi_S\sigma + (1+z+g_K-g_S)\pi_F\sigma_P)\sigma_\beta'U_{x\beta}\\ &+x[\pi_S(\mu - r\mathbf{e}) + (1+z+g_K-g_S)\pi_F(\alpha - r\mathbf{e}_m) + (\beta + r - r_D)z - (\pi_S\sigma + (1+z\\ &+g_K-g_S)\pi_F\sigma_P)\sigma_Z']U_x - xcU_x + V(cx)\} + \frac{1}{2}\sigma_Z\sigma_Z'U_{xx} + \frac{1}{2}\sigma_\beta\sigma_\beta'U_{\beta\beta}\\ &-x\sigma_Z\sigma_\beta'U_{x\beta} + x[r + (\xi + r - r_D)g_K + \beta(1+g_K) - \mu_I + \sigma_Z\sigma_Z']U_x\\ &+[f(\beta) + \beta\eta(\omega)]U_\beta.\end{aligned} \tag{50}$$

Using (49) on (50), we have the following:

$$\begin{aligned}0 = &\max_{z,\pi_S,\pi_F,c}\{-\frac{1}{2}\pi_S\sigma(\pi_S\sigma)'\gamma(1-\gamma)U(x,\beta) - (1+z+g_K\\ &-g_S)\pi_F\sigma_P(\pi_S\sigma)'\gamma(1-\gamma)U(x,\beta) - \frac{1}{2}(1+z+g_K-g_S)^2\pi_F\sigma_P(\pi_F\sigma_P)'\\ &\times\gamma(1-\gamma)U(x,\beta) + \pi_S\sigma\sigma_Z'\gamma(1-\gamma)U(x,\beta) - (1+z+g_K-g_S)\pi_F\sigma_P\sigma_Z'\gamma(1-\gamma)U(x,\beta)\\ &+h_\beta(\pi_S\sigma + (1+z+g_K-g_S)\pi_F\sigma_P)\sigma_\beta'(1-\gamma)U(x,\beta) + [\pi_S(\mu - r\mathbf{e})\\ &+(1+z+g_K-g_S)\pi_F(\alpha - r\mathbf{e}_m) + \beta z - (\pi_S\sigma + (1+z+g_K-g_S)\pi_F\sigma_P)\\ &\times\sigma_Z'](1-\gamma)U(x,\beta) - c(1-\gamma)U(x,\beta) + V(cx)\} - \frac{1}{2}\sigma_Z\sigma_Z'\gamma(1-\gamma)U(x,\beta)\\ &+\frac{1}{2}\sigma_\beta\sigma_\beta'[(h_\beta)^2 + h_{\beta\beta}]U(x,\beta) - h_\beta\sigma_Z\sigma_\beta'(1-\gamma)U(x,\beta) + [r + z(r - r_D) + (\xi + r - r_D)g_K\\ &+\beta(1+g_K) - \mu_I + \sigma_Z\sigma_Z'](1-\gamma)U(x,\beta) + [f(\beta) + \beta\eta(\omega)]h_\beta U(x,\beta) - \delta U(x,\beta).\end{aligned} \tag{51}$$

Now, dividing (51) through by $-(1-\gamma)U(x,\beta) < 0$, so that max becomes min, we have

$$\begin{aligned}0 = &\min_{z,\pi_S,\pi_F,c}\{\frac{1}{2}\pi_S\sigma(\pi_S\sigma)'\gamma + (1+z+g_K-g_S)\pi_F\sigma_P(\pi_S\sigma)'\gamma\\ &+\frac{1}{2}(1+z+g_K-g_S)^2\pi_F\sigma_P(\pi_F\sigma_P)'\gamma - \pi_S\sigma\sigma_Z'\gamma + (1+z+g_K-g_S)\pi_F\sigma_P\sigma_Z'\gamma\\ &-h_\beta(\pi_S\sigma + (1+z+g_K-g_S)\pi_F\sigma_P)\sigma_\beta' - [\pi_S(\mu - r\mathbf{e}) + (1+z+g_K-g_S)\pi_F(\alpha - r\mathbf{e}_m)\\ &+(\beta + r - r_D)z - (\pi_S\sigma + (1+z+g_K-g_S)\pi_F\sigma_P)\sigma_Z'] + c - \frac{V(cx)}{(1-\gamma)U(x,\beta)}\} + \frac{1}{2}\sigma_Z\sigma_Z'\gamma\\ &-\frac{1}{2(1-\gamma)}\sigma_\beta\sigma_\beta'[(h_\beta)^2 + h_{\beta\beta}] + h_\beta\sigma_Z\sigma_\beta' - [r + (\xi + r - r_D)g_K\\ &+\beta(1+g_K) - \mu_I + \sigma_Z\sigma_Z'] - \frac{1}{1-\gamma}[f(\beta) + \beta\eta(\omega)]h_\beta + \frac{\delta}{1-\gamma}.\end{aligned} \tag{52}$$

From (52), we deduce our optimal investment strategies for both FM and FA, optimal net debt ratio and optimal consumption strategy of the investor.

### 6.4. Optimal stocks investment policies

**Definition 8.** The optimal stocks investment policy with the portfolio weights strategy $\pi_S$ is defined as

$$\pi_S^{'*} = arg\ \min_{\pi_S} f_1(\pi_S), \tag{53}$$

where the function

$$\begin{aligned} f_1(\pi_S) &= \frac{1}{2}\pi_S\sigma(\pi_S\sigma)'\gamma + (1+z+g_K-g_S)\pi_F\sigma_P(\pi_S\sigma)'\gamma \\ &-\pi_S\sigma\sigma_Z'\gamma - h_\beta\pi_S\sigma\sigma_\beta' - [\pi_S(\mu - r\mathbf{e}) - \pi_S\sigma\sigma_Z'] \end{aligned} \tag{54}$$

is convex.

**Proposition 5.** *The optimal stocks investment strategies of the investor are*

$$\begin{aligned} \pi_S^{'*} &= \frac{1}{\gamma}\Sigma^{-1}(\mu - r\mathbf{e}) + \frac{\gamma-1}{\gamma}\Sigma^{-1}(\sigma\sigma_Z') + \frac{1}{\gamma}h_\beta\Sigma^{-1}\sigma\sigma_\beta' \\ &-(1+z^*+g_K-g_S)\Sigma^{-1}\sigma(\pi_F^*\sigma_P)'. \end{aligned} \tag{55}$$

**Proof.** By the first order conditions for the portfolio process, we have that

$$\begin{aligned} \frac{\partial f_1(\pi_S)}{\partial \pi_S} &= \sigma(\pi_S\sigma)'\gamma + (1+z+g_K-g_S)\sigma(\pi_F\sigma_P)'\gamma \\ &-\sigma\sigma_Z'\gamma - h_\beta\sigma\sigma_\beta' - (\mu - r\mathbf{e}) + \sigma\sigma_Z' = 0. \end{aligned} \tag{56}$$

Making $\pi_S^*$ the subject of the formula, we have

$$\begin{aligned} \pi_S^{'*} &= \frac{1}{\gamma}\underbrace{\Sigma^{-1}(\mu - r\mathbf{e})}_{\phi_1} + \frac{1}{\gamma}\underbrace{h_\beta\Sigma^{-1}\sigma\sigma_\beta'}_{\phi_2} - \underbrace{(1+z^*+g_K-g_S)\Sigma^{-1}\sigma(\pi_F^*\sigma_P)'}_{\phi_3} \\ &+\frac{\gamma-1}{\gamma}\underbrace{\Sigma^{-1}(\sigma\sigma_Z')}_{\phi_4}. \quad \square \end{aligned} \tag{57}$$

It shows that it is optimal to invest in a portfolio that comprises of four components:

1. a speculative portfolio $\phi_1$ proportional to the market price of risk corresponding to the risky assets through the relative risk averse index $\frac{1}{\gamma}$,
2. an income risk hedging portfolio $\phi_2$ proportional to the diffusion term of the income process through the cross derivative of function $h$ with respect to the income rate $\beta$ and the relative risk averse index $\frac{1}{\gamma}$,
3. a hedging portfolio against debt risk $\phi_3$ proportional to the diffusion term of the FA price and portfolio strategies of FA through the net debt ratio $z$, $g_S$ and $g_K$,
4. an inflation risk hedging portfolio $\phi_4$ proportional to the diffusion term of the consumer price index and the relative risk averse index $\frac{\gamma-1}{\gamma}$.

(57) is the investor's optimal portfolio strategies in the stock market with diffusion process. Our optimal stocks investment policies have a myopic and intertemporal hedging terms. The myopic demand, $\phi_1$ would be the investment policy for an investor who optimized over time and account for her future investments, or the optimal policy if the investment opportunities set were time dependent. The intertemporal hedging demands, $\phi_2$, $\phi_3$ and $\phi_4$ then, arise due to the need to hedge against the fluctuations in income, credit, inflation and the investment opportunities over time.

### 6.5. Optimal investment policies for FA

**Definition 9.** The optimal FA policy with the portfolio weights strategy $\pi_F$ is defined as

$$\pi_F^{'*} = arg\ \min_{\pi_F} f_2(\pi_F), \tag{58}$$

where the function

$$\begin{aligned} f_2(\pi_F) &= (1+z+g_K-g_S)\pi_F\sigma_P(\pi_S\sigma)'\gamma + \frac{1}{2}(1+z+g_K-g_S)^2\pi_F\sigma_P(\pi_F\sigma_P)'\gamma \\ &-(1+z+g_K-g_S)\pi_F\sigma_P\sigma_Z'\gamma - h_\beta(1+z+g_K-g_S)\pi_F\sigma_P\sigma_\beta' \\ &-[(1+z+g_K-g_S)\pi_F(\alpha - r\mathbf{e}_m) - (1+z+g_K-g_S)\pi_F\sigma_P\sigma_Z'] \end{aligned} \tag{59}$$

is convex.

**Proposition 6.** *The optimal FA investment strategies of the investor are*

$$\begin{aligned}\pi_F^{'*} &= \frac{\Sigma_P{}^{-1}(\alpha - r\mathbf{e}_m)}{\gamma(1+z^*+g_K-g_S)} - \frac{\Sigma_P{}^{-1}\sigma_P(\pi_S^*\sigma)'}{1+z^*+g_K-g_S} + \frac{h_\beta \Sigma_P{}^{-1}\sigma_P\sigma_\beta'}{\gamma(1+z^*+g_K-g_S)} \\ &- \frac{(1-\gamma)\Sigma_P{}^{-1}\sigma_P\sigma_Z'}{\gamma(1+z^*+g_K-g_S)}.\end{aligned} \tag{60}$$

**Proof.** By the first order conditions for the portfolio process, we have that

$$\begin{aligned}&\frac{\partial f_2(\pi_F)}{\partial \pi_F} = \sigma_P(\pi_S\sigma)'\gamma + (1+z+g_K-g_S)\sigma_P(\pi_F\sigma_P)'\gamma \\ &- h_\beta\sigma_P\sigma_\beta' - (\alpha - r\mathbf{e}_m) + (1-\gamma)\sigma_P\sigma_Z' = 0.\end{aligned} \tag{61}$$

Making $\pi_F^*$ the subject of the formula, we have

$$\begin{aligned}\pi_F^{'*} &= \frac{1}{\gamma}\underbrace{\frac{\Sigma_P{}^{-1}(\alpha - r\mathbf{e}_m)}{1+z^*+g_K-g_S}}_{\psi_1} - \underbrace{\frac{\Sigma_P{}^{-1}\sigma_P(\pi_S^*\sigma)'}{1+z^*+g_K-g_S}}_{\psi_2} + \frac{1}{\gamma}\underbrace{\frac{h_\beta \Sigma_P{}^{-1}\sigma_P\sigma_\beta'}{1+z^*+g_K-g_S}}_{\psi_3} \\ &- \frac{(1-\gamma)}{\gamma}\underbrace{\frac{\Sigma_P{}^{-1}\sigma_P\sigma_Z'}{1+z^*+g_K-g_S}}_{\psi_4}. \quad \square\end{aligned} \tag{62}$$

It shows that it is optimal to invest in a FA portfolio that comprises of four components:

1. a speculative portfolio $\psi_1$ proportional to the market price of risk corresponding to the FAs through the inverse of the term $1+z^*+g_K-g_S$ and the relative risk averse index $\frac{1}{\gamma}$,
2. a hedging portfolio against debt risk $\psi_2$ proportional to the diffusion term of the FA price and portfolio strategies in stocks through the inverse of the term $1+z^*+g_K-g_S$,
3. an income risk hedging portfolio $\psi_3$ proportional to the diffusion term of the income process through the inverse of the term $1+z^*+g_K-g_S$ and the cross derivative of function $h$ with respect to the income rate $\beta$ and the relative risk averse index $\frac{1}{\gamma}$,
4. an inflation risk hedging portfolio $\psi_4$ proportional to the diffusion term of the consumer price index through the inverse of the term $1+z^*+g_K-g_S$ and the relative risk averse index $\frac{\gamma-1}{\gamma}$.

(62) is the investor's optimal portfolio strategies in the FA market. It is observed that the investment in FA by an investor will lead to a total hedging of credit risks in the stock market portfolio process.

### 6.6. Optimal net debt ratio policy

**Definition 10.** The optimal policy for the net debt ratio $z$ is

$$z^* = arg\min_z f_3(z), \tag{63}$$

where the function

$$\begin{aligned}f_3(z) &= z\pi_F\sigma_P(\pi_S\sigma)'\gamma + \frac{1}{2}(2z + 2zg_K - 2zg_S + z^2)\pi_F\sigma_P(\pi_F\sigma_P)'\gamma \\ &- z\pi_F\sigma_P\sigma_Z'\gamma - h_\beta z\pi_F\sigma_P\sigma_\beta' - z\pi_F(\alpha - r\mathbf{e}_m) - (\beta + r - r_D)z + z\pi_F\sigma_P\sigma_Z'.\end{aligned} \tag{64}$$

is convex.

**Proposition 7.** *The optimal net debt ratio of the investor is*

$$\begin{aligned}z^* &= -(1+g_K-g_S) - \frac{\pi_F^*\sigma_P(\pi_S^*\sigma)'}{\pi_F^*\sigma_P(\pi_F^*\sigma_P)'} + \frac{1}{\gamma}\frac{h_\beta\pi_F^*\sigma_P\sigma_\beta'}{\pi_F^*\sigma_P(\pi_F^*\sigma_P)'} \\ &+ \frac{1}{\gamma}\frac{\pi_F^*(\alpha - r\mathbf{e}_m)}{\pi_F^*\sigma_P(\pi_F^*\sigma_P)'} + \frac{1}{\gamma}\frac{\beta + r - r_D}{\pi_F^*\sigma_P(\pi_F^*\sigma_P)'} + \frac{\gamma - 1}{\gamma}\frac{\pi_F^*\sigma_P\sigma_Z'}{\pi_F^*\sigma_P(\pi_F^*\sigma_P)'}.\end{aligned} \tag{65}$$

**Proof.** By the principles of the first order conditions, we have that

$$\begin{aligned}&\frac{\partial f_3(z)}{\partial z} = \pi_F\sigma_P(\pi_S\sigma)'\gamma + (1+g_K-g_S+z)\pi_F\sigma_P(\pi_F\sigma_P)'\gamma \\ &- \pi_F\sigma_P\sigma_Z'\gamma - h_\beta\pi_F\sigma_P\sigma_\beta' - \pi_F(\alpha - r\mathbf{e}_m) - (\beta + r - r_D) + \pi_F\sigma_P\sigma_Z' = 0.\end{aligned} \tag{66}$$

Making $z$ the subject, we have

$$\begin{aligned} z^* &= -(1+g_K-g_S) - \frac{\pi_F^*\sigma_P(\pi_S^*\sigma)'}{\pi_F^*\sigma_P(\pi_F^*\sigma_P)'} + \frac{1}{\gamma}\frac{h_\beta\pi_F^*\sigma_P\sigma_\beta'}{\pi_F^*\sigma_P(\pi_F^*\sigma_P)'} \\ &+ \frac{1}{\gamma}\frac{\pi_F^*(\alpha-r\mathbf{e}_m)}{\pi_F^*\sigma_P(\pi_F^*\sigma_P)'} + \frac{1}{\gamma}\frac{\beta+r-r_D}{\pi_F^*\sigma_P(\pi_F^*\sigma_P)'} + \frac{\gamma-1}{\gamma}\frac{\pi_F^*\sigma_P\sigma_Z'}{\pi_F^*\sigma_P(\pi_F^*\sigma_P)'}. \quad \square \end{aligned} \tag{67}$$

(67) is the optimal debt ratio for an investor. It is observed that the ratio $g_K$ will play a vital role in minimizing the debt risk in the stock and FA market investment portfolios.

**Corollary 1.** *Suppose an investor is free from debt, then*

$$\begin{aligned} \pi_S^{'*} &= \frac{1}{\gamma}\Sigma^{-1}(\mu-r\mathbf{e}) + \frac{\gamma-1}{\gamma}\Sigma^{-1}(\sigma\sigma_Z') + \frac{1}{\gamma}h_\beta\Sigma^{-1}\sigma\sigma_\beta' \\ &-(1+g_K-g_S)\Sigma^{-1}\sigma(\pi_F^*\sigma_P)', \end{aligned}$$

$$\begin{aligned} \pi_F^{'*} &= \frac{{\Sigma_P}^{-1}(\alpha-r\mathbf{e}_m)}{\gamma(1+g_K-g_S)} - \frac{{\Sigma_P}^{-1}\sigma_P(\pi_S^*\sigma)'}{1+g_K-g_S} + \frac{h_\beta{\Sigma_P}^{-1}\sigma_P\sigma_\beta'}{\gamma(1+g_K-g_S)} \\ &- \frac{(1-\gamma){\Sigma_P}^{-1}\sigma_P\sigma_Z'}{\gamma(1+g_K-g_S)}. \end{aligned}$$

*and*

$$\begin{aligned} g_K &= -(1-g_S) - \frac{\pi_F\sigma_P(\pi_S^*\sigma)'}{\pi_F^*\sigma_P(\pi_F^*\sigma_P)'} + \frac{1}{\gamma}\frac{h_\beta\pi_F^*\sigma_P\sigma_\beta'}{\pi_F^*\sigma_P(\pi_F^*\sigma_P)'} \\ &+ \frac{1}{\gamma}\frac{\pi_F^*(\alpha-r\mathbf{e}_m)}{\pi_F^*\sigma_P(\pi_F^*\sigma_P)'} + \frac{1}{\gamma}\frac{\beta+r-r_D}{\pi_F^*\sigma_P(\pi_F^*\sigma_P)'} + \frac{\gamma-1}{\gamma}\frac{\pi_F^*\sigma_P\sigma_Z'}{\pi_F^*\sigma_P(\pi_F^*\sigma_P)'}. \end{aligned}$$

### *6.7. Optimal consumption policy*

In this subsection, we give the optimal consumption plan of the investor. For a given wealth $x$, the optimal consumption choice is therefore obtained as

$$c^*(t) = \left[\frac{\partial V}{\partial c}\right]^{-1}(xU_x). \tag{68}$$

Hence, the optimal consumption plan $c^*$ is therefore obtained as

$$c^* = \left(\frac{\partial V}{\partial c}\right)^{-1}((1-\gamma)U(x,\beta)). \tag{69}$$

We now have the following proposition.

**Proposition 8.** *The optimal consumption plan of the investor is*

$$c^* = e^{-\frac{h(\beta)}{\gamma}}.$$

**Proof.** For the optimal consumption policy, with $\left[\frac{\partial V}{\partial c}\right]^{-1}(y) = y^{-\frac{1}{\gamma}}$ and $(1-\gamma)U(x,\beta) = x^{1-\gamma}e^{h(\beta)}$ in (69), we obtain the following:

$$c^* = ((\partial U/\partial c)^{-1})((1-\gamma)U(x,\beta)) = e^{-\frac{h(\beta)}{\gamma}}. \quad \square \tag{70}$$

It is observed that if $h(\beta) > 0$, consumption rate will increase as $\gamma$ decreases and decrease as $\gamma$ increases. We also observe that as $h(\beta)$ becomes very large for all other parameters remain fixed, consumption tends to zero and consumption becomes very large as $h(\beta)$ tends to zero. Clearly, if $h(\beta) < 0$, consumption will continue to increase as $h(\beta)$ increases. If $\gamma \longrightarrow +\infty$ and $h(\beta) \longrightarrow -\infty$, $c^* \longrightarrow 0$ and vice versa.

## 7. Optimal investments in terms of $z^*$ and the explicit form of our HJB equation

In this section, we give the optimal investment in FM and FA in terms of $z^*$ and the explicit form of our HJB equation.

### 7.1. Optimal investment in FM and FA in terms of $z^*$

From (55) and (60), we have the following:

$$\pi_S^{'*} = D_1 + \nu_1 h_\beta - \nu_2 \pi_F^{*'}, \tag{71}$$

$$\pi_F^{'*} = D_2 + \rho_1 h_\beta - \rho_2 \pi_S^{*'}, \tag{72}$$

where
$D_1 = \frac{1}{\gamma}\Sigma^{-1}(\mu - r\mathbf{e}) + \frac{\gamma-1}{\gamma}\Sigma^{-1}(\sigma\sigma_Z'), \nu_1 = \frac{1}{\gamma}\Sigma^{-1}\sigma\sigma_\beta', \nu_2 = \varphi\Sigma^{-1}(\sigma\sigma_P'), D_2 = \frac{\Sigma_P{}^{-1}(\alpha - r\mathbf{e}_m)}{\gamma\varphi} - \frac{(1-\gamma)\Sigma_P{}^{-1}\sigma_P\sigma_Z'}{\gamma\varphi}, \rho_1 = \frac{\Sigma_P{}^{-1}\sigma_P\sigma_\beta'}{\gamma\varphi}$, $\rho_2 = \frac{\Sigma_P{}^{-1}(\sigma_P\sigma')}{\varphi}, \varphi = 1 + z^* + g_K - g_S, \mathbf{1} \in \mathcal{R}^{n\times n}$ identity matrix.

Using (71) and (72), we have

$$\pi_S^{'*} + \nu_2\pi_F^{*'} = D_1 + \nu_1 h_\beta. \tag{73}$$

$$\rho_2\pi_S^{*'} + \pi_F^{'*} = D_2 + \rho_1 h_\beta. \tag{74}$$

From (73) and (74), we have the following:

$$\begin{pmatrix} \mathbf{1} & \nu_2 \\ \rho_2 & \mathbf{1} \end{pmatrix}\begin{pmatrix} \pi_S^{'*} \\ \pi_F^{'*} \end{pmatrix} = \begin{pmatrix} D_1 + \nu_1 h_\beta \\ D_2 + \rho_1 h_\beta \end{pmatrix}. \tag{75}$$

Solving (75) simultaneously, we have that

$$\pi_S^{'*} = J^{-1}[D_1 - \nu_2 D_2 + (\nu_1 - \nu_2\rho_1)h_\beta], \tag{76}$$

$$\pi_F^{'*} = J^{-1}[JD_2 - \rho_2 D_1 + \rho_2\nu_2 D_2 + (J\rho_1 - \rho_2(\nu_1 - \nu_2\rho_1))h_\beta], \tag{77}$$

where $J = \mathbf{1} - \nu_2\rho_2$.

### 7.2. Explicit form of the HJB equation

We now give the explicit form of our HJB equation in this subsection. First, we re-write (76) and (77) as follows:

$$\pi_S^{'*} = a_1 + b_1 h_\beta, \tag{78}$$

$$\pi_F^{'*} = a_2 + b_2 h_\beta, \tag{79}$$

where $a_1 = J^{-1}[D_1 - \nu_2 D_2]$, $b_1 = J^{-1}[\nu_1 - \nu_2\rho_1]$, $a_2 = J^{-1}[JD_2 - \rho_2 D_1 + \rho_2\nu_2 D_2]$, $b_2 = J^{-1}[J\rho_1 - \rho_2(\nu_1 - \nu_2\rho_1)]$.

**Proposition 9.** *The explicit form of our HJB equation (52) is of the form*

$$\begin{aligned}(\eta + \chi_1)h_{\beta\beta} &+ \left(\chi_2 - \frac{1}{1-\gamma}[f(\beta) + \beta\eta(\omega)]\right)h_\beta + e^{-\frac{h(\beta)}{\gamma}} \\ &- \beta(\varphi + g_S) + \chi_3 = 0,\end{aligned} \tag{80}$$

*where*

$$\eta = -\frac{1}{2(1-\gamma)}\sigma_\beta\sigma_\beta',$$
$$\chi_1 = \frac{\gamma}{2}b_1 a_1\sigma b_1'\sigma' + \gamma\varphi b_2\sigma_P b_1'\sigma' + \frac{1}{2}\varphi^2\gamma b_2\sigma_P b_2'\sigma_P' - b_1\sigma\sigma_\beta' - \varphi b_2\sigma_P\sigma_\beta' + \eta,$$

$$\begin{aligned}\chi_2 &= \gamma a_1\sigma b_1'\sigma' + \gamma\varphi(b_2\sigma_P a_1'\sigma' + a_2\sigma_P b_1'\sigma') + \varphi^2\gamma b_2\sigma_P a_2'\sigma_P' - b_1\sigma\sigma_Z'\gamma \\ &+ \varphi b_2\sigma_P\sigma_Z'\gamma - a_1\sigma\sigma_\beta' - \varphi a_2\sigma_P\sigma_\beta' - b_1(\mu - r\mathbf{e}) - \varphi b_2(\alpha - r\mathbf{e}_m) + b_1\sigma\sigma_Z' + \varphi b_2\sigma_P\sigma_Z' \\ &+ \sigma_Z\sigma_\beta',\end{aligned}$$

$$\begin{aligned}\chi_3 &= \frac{\gamma}{2}a_1\sigma a_1'\sigma' + \gamma\varphi a_2\sigma_P a_1'\sigma' + \frac{1}{2}\varphi^2\gamma a_2\sigma_P a_2'\sigma_P' - a_1\sigma\sigma_Z'\gamma + \varphi a_2\sigma_P\sigma_Z'\gamma \\ &- a_1(\mu - r\mathbf{e}) - \varphi a_2(\alpha - r\mathbf{e}_m) + a_1\sigma\sigma_Z' + \varphi a_2\sigma_P\sigma_Z' + \frac{1}{2}\sigma_Z\sigma_Z'\gamma \\ &- (r + (\xi + r - r_D)g_K)(1 - \varphi) + \mu_I - \sigma_Z\sigma_Z' + \frac{\delta}{1-\gamma}.\end{aligned}$$

**Table 1**
Stocks and fixed assets information.

| Stock | Mean price | Slope | SD | FA | Mean price | Slope | SD |
|---|---|---|---|---|---|---|---|
| 7Up | 96.441 | 0.0001 | 0.793201 | FA1 | 435.5776 | 0.0001 | 8.294039 |
| AB | 7.88222 | 0.0018 | 0.640534 | FA2 | 836.1396 | 0.0001 | 7.906477 |
| GTB | 27.656 | 0.0012 | 0.690393 | FA3 | 419.4286 | 0.0001 | 6.014989 |
| COiL | 34.967 | 0.0025 | 0.894483 | FA4 | 606.3917 | 0.0001 | 1.572066 |
| Oando | 8.2225 | 0.0002 | 0.611450 | FA5 | 742.2376 | 0.0001 | 7.44897 |
| DB | 1.07556 | 0.0004 | 0.125935 | FA6 | 277.0427 | 0.0001 | 6.331878 |
| PZ | 16.888 | 0.0013 | 0.723558 | FA7 | 254.8473 | 0.0001 | 2.590696 |
| UBA | 7.22694 | 0.0003 | 0.581460 | FA8 | 954.8444 | 0.0001 | 2.622017 |
| ZB | 17.237 | 0.0004 | 0.672061 | FA9 | 735.3981 | 0.0001 | 8.168191 |
| WB | 1.48778 | 0.0001 | 0.551647 | FA10 | 177.0101 | 0.0001 | 6.328203 |

**Proof.** Substituting (78) and (79) into (52) and then simplify, we have

$$\begin{aligned}&\eta h_{\beta\beta} + \chi_1(h_\beta)^2 + \left(\chi_2 - \frac{1}{1-\gamma}[f(\beta) + \beta\eta(\omega)]\right) h_\beta + e^{-\frac{h(\beta)}{\gamma}}\\&-\beta(\varphi + g_S) + \chi_3 = 0.\end{aligned} \tag{81}$$

Solving the following ODE $h_{\beta\beta} = (h_\beta)^2$, we have a solution of the form:

$$\begin{cases} h(\beta(t)) = G_1(t) - \ln[G_2(t) + E_1(t)\beta] \\ h(\beta(T)) = 0, \end{cases} \tag{82}$$

where $G_1(t), G_2(t), E_1(t) \in \mathcal{R}$ and $G_1(t) + E_1(t)\beta > 0$. (81) now becomes

$$\begin{aligned}&(\eta + \chi_1)h_{\beta\beta} + \left(\chi_2 - \frac{1}{1-\gamma}[f(\beta) + \beta\eta(\omega)]\right) h_\beta + e^{-\frac{h(\beta)}{\gamma}}\\&-\beta(\varphi + g_S) + \chi_3 = 0. \quad \square\end{aligned} \tag{83}$$

Since $(\gamma, \mu, r, \Sigma)$ are all not constants (where $\mu(t)$, $\Sigma(t)$ and $r(t)$ are time-dependent), the objective functions $f_1, f_2$ and $f_3$ are time-dependent. Hence, the optimal solutions will indeed be time-dependent. Again, the objective functions are state dependent as well. It then follows that any optimal solution will be state dependent. In other words, any optimal portfolio strategies $\pi_S^{'*}(t)$, $\pi_F^{'*}(t)$ and the debt ratio $z^*$ will be time-dependent and it follows that $\pi_S^{'*}(t)$, $\pi_F^{'*}(t)$ and $z^*$ are dependent of time and state variables. Finally, the objective functions $f_1, f_2$ and $f_3$ are strictly convex, goes to $+\infty$ in all directions, so that unique minimizers will always be obtained.

Finally, we have to check to ensure that the transversality condition is satisfied. We do that by substituting the optimizers $x^*$ and $c^*$ into (40), and then taking expectations to have the following:

$$G(t, x^*(t), \beta(t)) = E_t\left[\int_t^\infty e^{-\delta s} V(c^*(s)x^*(s))ds\right]. \tag{84}$$

Taking the limit as $t$ tends to infinity, there will be exponential decay of (84). Substituting $\pi_S^{'*}$, $\pi_F^{'*}$, $z^*$ and $c^*$ into (29), we found that an investor with power utility who select these optimal portfolio in stocks, optimal portfolio in FAs, optimal debt ratio and consumption plan will achieve a wealth process $x^*(t) = \bar{X}^*(t)$ that follows a geometric Brownian process.

## 8. Empirical results

In this section, we give the empirical results of our problem. Data used in this work are obtained from National Bureau of Statistics, International Monetary Fund and Nigeria Stock Exchange. The inflation data from the International Monetary Fund from 1980 to June 18, 2017 were collected and analyzed. Stock prices of ten companies were collected from Nigeria Stock Exchange from 2012 to May, 2017. The companies stock prices used in this work are Seven-Up Bottling Company PLC (7UP), Access Bank of Nigeria PLC (AB), Guarantee Trust Bank PLC (GTB), CONOIL (CoiL), Oando, Diamond Bank of Nigeria PLC (DB), PZ, United Bank for Africa PLC (UBA), Zenith Bank of Nigeria PLC (ZB) and Wema Bank of Nigeria PLC (WB). The data collected from these companies were analyzed using Statistical Package for Social Sciences (SPSS). Table 1 show the mean, standard deviation (SD) and the slope (SP) (or growth rate of the stock and FA over time) of the stocks and FAs. The growth rate (GR) of the stock and FA prices is assumed to be time dependent and follows a linear growth rate with intercept taken to be the mean price and slope (SP). Also, obtained are the variance–covariance matrices of stocks, inflation, income growth rate and FA, and are given below. The resulting optimization problem is solved using MatLab 7.5.0 and the results are presented in Tables 2–5.

**Table 2**
Optimal portfolio returns from the FA for varying $z^*$ and $\gamma$, and $r = 0.02$.

| $z^*$ | $\gamma$ | FA1 | FA2 | FA3 | FA4 | FA5 | FA6 | FA7 | FA8 | FA9 | FA10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.2 | 0.1 | 354.7106 | 375.9951 | 382.1107 | 393.0764 | 381.8045 | 107.2004 | 332.7789 | 445.5595 | 392.6691 | 336.4476 |
| | 0.2 | 177.4067 | 188.0480 | 191.0983 | 196.5770 | 190.9512 | 53.7255 | 166.4429 | 222.8229 | 196.3821 | 168.3121 |
| | 0.3 | 118.3054 | 125.3990 | 127.4275 | 131.0772 | 127.3335 | 35.9006 | 110.9975 | 148.5774 | 130.9531 | 112.2669 |
| | 0.4 | 88.7547 | 94.0745 | 95.5921 | 98.3272 | 95.5246 | 26.9881 | 83.2749 | 111.4546 | 98.2385 | 84.2443 |
| | 0.5 | 71.0243 | 75.2797 | 76.4909 | 78.6773 | 76.4393 | 21.6406 | 66.6413 | 89.1810 | 78.6098 | 67.4307 |
| | 1.0 | 35.5635 | 37.6903 | 38.2884 | 39.3774 | 38.2686 | 10.9457 | 33.3741 | 44.6336 | 39.3524 | 33.8036 |
| | 2.0 | 17.8331 | 18.8956 | 19.1872 | 19.7275 | 19.1833 | 5.5982 | 16.7405 | 22.3600 | 19.7237 | 16.9901 |
| | 3.0 | 11.9230 | 12.6307 | 12.8201 | 13.1775 | 12.8215 | 3.8157 | 11.1959 | 14.9354 | 13.1808 | 11.3856 |
| | 4.0 | 8.9679 | 9.4983 | 9.6366 | 9.9025 | 9.6407 | 2.9244 | 8.4237 | 11.2232 | 9.9094 | 8.5833 |
| | 5.0 | 7.1949 | 7.6188 | 7.7264 | 7.9375 | 7.7321 | 2.3897 | 6.7603 | 8.9958 | 7.9465 | 6.9019 |
| | 10.0 | 3.6488 | 3.8598 | 3.9062 | 4.0075 | 3.9151 | 1.3202 | 3.4336 | 4.5411 | 4.0208 | 3.5392 |
| | 50.0 | 0.8120 | 0.8527 | 0.8500 | 0.8635 | 0.8614 | 0.4646 | 0.7722 | 0.9773 | 0.8802 | 0.8491 |
| 0.6 | 0.1 | 269.8083 | 285.9982 | 290.6500 | 298.9910 | 290.4171 | 81.5413 | 253.1261 | 338.9119 | 298.6812 | 255.9167 |
| | 0.2 | 134.9432 | 143.0375 | 145.3577 | 149.5250 | 145.2458 | 40.8660 | 126.6037 | 169.4888 | 149.3767 | 128.0255 |
| | 0.3 | 89.9882 | 95.3839 | 96.9269 | 99.7030 | 96.8554 | 27.3075 | 84.4296 | 113.0144 | 99.6086 | 85.3951 |
| | 0.4 | 67.5107 | 71.5571 | 72.7115 | 74.7920 | 72.6602 | 20.5283 | 63.3425 | 84.7772 | 74.7245 | 64.0799 |
| | 0.5 | 54.0242 | 57.2610 | 58.1823 | 59.8454 | 58.1430 | 16.4608 | 50.6902 | 67.8349 | 59.7941 | 51.2907 |
| | 1.0 | 27.0512 | 28.6689 | 29.1238 | 29.9522 | 29.1088 | 8.3257 | 25.3858 | 33.9503 | 29.9332 | 25.7125 |
| | 2.0 | 13.5647 | 14.3728 | 14.5946 | 15.0056 | 14.5917 | 4.2582 | 12.7335 | 17.0080 | 15.0027 | 12.9234 |
| | 3.0 | 9.0692 | 9.6075 | 9.7515 | 10.0234 | 9.7526 | 2.9024 | 8.5161 | 11.3605 | 10.0259 | 8.6603 |
| | 4.0 | 6.8214 | 7.2248 | 7.3300 | 7.5323 | 7.3331 | 2.2244 | 6.4074 | 8.5368 | 7.5375 | 6.5288 |
| | 5.0 | 5.4728 | 5.7952 | 5.8771 | 6.0376 | 5.8814 | 1.8177 | 5.1422 | 6.8426 | 6.0445 | 5.2499 |
| | 10.0 | 2.7755 | 2.9360 | 2.9712 | 3.0483 | 2.9780 | 1.0042 | 2.6117 | 3.4541 | 3.0584 | 2.6921 |
| | 50.0 | 0.6176 | 0.6486 | 0.6465 | 0.6568 | 0.6552 | 0.3534 | 0.5874 | 0.7434 | 0.6695 | 0.6458 |
| 1.0 | 0.1 | 215.3290 | 228.4066 | 232.1922 | 238.9184 | 231.9478 | 62.8824 | 201.8508 | 271.0794 | 238.6160 | 203.9329 |
| | 0.2 | 107.6960 | 114.2342 | 116.1225 | 119.4830 | 116.0040 | 31.5181 | 100.9582 | 135.5662 | 119.3371 | 102.0206 |
| | 0.3 | 71.8184 | 76.1768 | 77.4326 | 79.6712 | 77.3560 | 21.0634 | 67.3273 | 90.3951 | 79.5775 | 68.0498 |
| | 0.4 | 53.8795 | 57.1481 | 58.0876 | 59.7653 | 58.0321 | 15.8360 | 50.5119 | 67.8096 | 59.6977 | 51.0645 |
| | 0.5 | 43.1163 | 45.7309 | 46.4806 | 47.8217 | 46.4377 | 12.6996 | 40.4226 | 54.2583 | 47.7698 | 40.8732 |
| | 1.0 | 21.5897 | 22.8964 | 23.2667 | 23.9347 | 23.2489 | 6.4267 | 20.2441 | 27.1556 | 23.9141 | 20.4908 |
| | 2.0 | 10.8264 | 11.4792 | 11.6597 | 11.9911 | 11.6545 | 3.2903 | 10.1548 | 13.6043 | 11.9862 | 10.2996 |
| | 3.0 | 7.2386 | 7.6734 | 7.7907 | 8.0099 | 7.7897 | 2.2448 | 6.7918 | 9.0872 | 8.0102 | 6.9025 |
| | 4.0 | 5.4447 | 5.7705 | 5.8562 | 6.0193 | 5.8573 | 1.7221 | 5.1102 | 6.8286 | 6.0223 | 5.2039 |
| | 5.0 | 4.3684 | 4.6288 | 4.6955 | 4.8250 | 4.6979 | 1.4084 | 4.1013 | 5.4735 | 4.8295 | 4.1848 |
| | 10.0 | 2.2157 | 2.3454 | 2.3741 | 2.4363 | 2.3790 | 0.7811 | 2.0834 | 2.7632 | 2.4439 | 2.1466 |
| | 50.0 | 0.4936 | 0.5186 | 0.5170 | 0.5253 | 0.5239 | 0.2793 | 0.4692 | 0.5950 | 0.5354 | 0.5160 |
| 5.0 | 0.1 | 73.4586 | 77.9200 | 79.2114 | 81.5060 | 79.1281 | 21.4521 | 68.8606 | 92.4776 | 81.4029 | 69.5709 |
| | 0.2 | 36.7401 | 38.9705 | 39.6147 | 40.7611 | 39.5743 | 10.7523 | 34.4415 | 46.2479 | 40.7114 | 34.8039 |
| | 0.3 | 24.5005 | 25.9874 | 26.4158 | 27.1795 | 26.3897 | 7.1857 | 22.9684 | 30.8379 | 27.1476 | 23.2149 |
| | 0.4 | 18.3808 | 19.4958 | 19.8164 | 20.3887 | 19.7974 | 5.4024 | 17.2319 | 23.1330 | 20.3656 | 17.4204 |
| | 0.5 | 14.7089 | 15.6009 | 15.8567 | 16.3142 | 15.8420 | 4.3324 | 13.7900 | 18.5100 | 16.2965 | 13.9437 |
| | 1.0 | 7.3652 | 7.8110 | 7.9373 | 8.1652 | 7.9313 | 2.1924 | 6.9062 | 9.2640 | 8.1582 | 6.9903 |
| | 2.0 | 3.6934 | 3.9161 | 3.9777 | 4.0907 | 3.9759 | 1.1225 | 3.4643 | 4.6411 | 4.0890 | 3.5137 |
| | 3.0 | 2.4694 | 2.6178 | 2.6578 | 2.7326 | 2.6574 | 0.7658 | 2.3170 | 3.1001 | 2.7327 | 2.3548 |
| | 4.0 | 1.8574 | 1.9686 | 1.9978 | 2.0535 | 1.9982 | 0.5875 | 1.7433 | 2.3296 | 2.0545 | 1.7753 |
| | 5.0 | 1.4903 | 1.5791 | 1.6019 | 1.6460 | 1.6027 | 0.4805 | 1.3991 | 1.8673 | 1.6476 | 1.4276 |
| | 10.0 | 0.7559 | 0.8001 | 0.8099 | 0.8311 | 0.8116 | 0.2665 | 0.7108 | 0.9427 | 0.8337 | 0.7323 |
| | 50.0 | 0.1684 | 0.1769 | 0.1764 | 0.1792 | 0.1787 | 0.0953 | 0.1600 | 0.2030 | 0.1827 | 0.1760 |

The covariance matrix $\sigma_{10}$ for the ten stocks is obtained as

$$\sigma_{10} = \begin{pmatrix} 0.038 & -0.004 & -0.005 & -0.001 & 0.003 & 0.033 & 0.011 & 0.007 & 0.014 & -0.021 \\ -0.004 & 0.046 & 0.003 & -0.003 & -0.005 & 0.024 & -0.002 & -0.017 & -0.005 & 0.011 \\ -0.005 & 0.003 & 0.050 & 0.003 & -0.008 & -0.029 & -0.006 & -0.017 & -0.019 & -0.015 \\ 0.000 & -0.003 & 0.003 & 0.023 & 0.003 & -0.035 & -0.007 & 0.004 & -0.007 & -0.003 \\ 0.003 & -0.005 & -0.008 & 0.003 & 0.050 & 0.017 & 0.002 & 0.010 & 0.003 & -0.016 \\ 0.033 & 0.024 & -0.029 & -0.035 & 0.017 & 1.171 & 0.018 & 0.013 & 0.040 & 0.052 \\ 0.011 & -0.002 & -0.006 & -0.007 & 0.002 & 0.018 & 0.038 & 0.004 & 0.005 & -0.016 \\ 0.007 & -0.017 & -0.017 & 0.004 & 0.010 & 0.013 & 0.004 & 0.064 & -0.004 & -0.008 \\ 0.014 & -0.005 & -0.019 & -0.007 & 0.003 & 0.040 & 0.005 & -0.004 & 0.052 & 0.008 \\ -0.021 & 0.011 & -0.015 & -0.003 & -0.016 & 0.052 & -0.016 & -0.008 & 0.008 & 0.087 \end{pmatrix}. \quad (85)$$

**Table 3**
The investor's optimal portfolio returns from the FM for varying $\gamma$ and $r$, and $z^* = 0.6$.

| $\gamma$ | $r$ | $\pi_{S0}$ | 7UP | AB | GTB | COiL | Oando | DB | PZ | UBA | ZB | WB |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.1 | 0.01 | 886.5689 | −72.1948 | −98.6601 | −97.3669 | −115.6064 | −99.5192 | −102.8616 | −25.3056 | −193.2542 | −128.1286 | 46.7284 |
| | 0.02 | 888.1770 | −72.3706 | −98.8368 | −97.5524 | −115.7923 | −99.6992 | −102.8321 | −25.4871 | −193.4307 | −128.3017 | 46.5260 |
| | 0.03 | 889.7850 | −72.5465 | −99.0135 | −97.7380 | −115.9782 | −99.8791 | −102.8027 | −25.6686 | −193.6072 | −128.4749 | 46.3235 |
| | 0.04 | 891.3931 | −72.7223 | −99.1903 | −97.9236 | −116.1641 | −100.0591 | −102.7732 | −25.8500 | −193.7836 | −128.6480 | 46.1211 |
| | 0.05 | 893.0012 | −72.8981 | −99.3670 | −98.1091 | −116.3500 | −100.2390 | −102.7438 | −26.0315 | −193.9601 | −128.8211 | 45.9186 |
| | 0.10 | 901.0416 | −73.7773 | −100.2507 | −99.0369 | −117.2794 | −101.1388 | −102.5965 | −26.9390 | −194.8426 | −129.6868 | 44.9064 |
| | 0.15 | 909.0819 | −74.6565 | −101.1344 | −99.9647 | −118.2088 | −102.0386 | −102.4492 | −27.8465 | −195.7250 | −130.5524 | 43.8942 |
| | 0.20 | 917.1223 | −75.5356 | −102.0181 | −100.8925 | −119.1382 | −102.9384 | −102.3020 | −28.7540 | −196.6074 | −131.4181 | 42.8820 |
| | 0.25 | 925.1627 | −76.4148 | −102.9018 | −101.8203 | −120.0676 | −103.8382 | −102.1547 | −29.6614 | −197.4899 | −132.2837 | 41.8697 |
| 0.5 | 0.01 | 177.1570 | −14.3921 | −19.6847 | −19.4107 | −23.0339 | −19.8522 | −20.4641 | −5.0187 | −38.6115 | −25.5633 | 9.2742 |
| | 0.02 | 177.4787 | −14.4273 | −19.7200 | −19.4478 | −23.0711 | −19.8882 | −20.4583 | −5.0550 | −38.6468 | −25.5980 | 9.2337 |
| | 0.03 | 177.8003 | −14.4625 | −19.7554 | −19.4849 | −23.1082 | −19.9242 | −20.4524 | −5.0913 | −38.6821 | −25.6326 | 9.1933 |
| | 0.04 | 178.1219 | −14.4976 | −19.7907 | −19.5220 | −23.1454 | −19.9602 | −20.4465 | −5.1276 | −38.7174 | −25.6672 | 9.1528 |
| | 0.05 | 178.4435 | −14.5328 | −19.8261 | −19.5591 | −23.1826 | −19.9961 | −20.4406 | −5.1639 | −38.7527 | −25.7018 | 9.1123 |
| | 0.10 | 180.0516 | −14.7086 | −20.0028 | −19.7447 | −23.3685 | −20.1761 | −20.4111 | −5.3454 | −38.9292 | −25.8750 | 8.9098 |
| | 0.15 | 181.6597 | −14.8845 | −20.1796 | −19.9303 | −23.5544 | −20.3561 | −20.3817 | −5.5269 | −39.1057 | −26.0481 | 8.7074 |
| | 0.20 | 183.2677 | −15.0603 | −20.3563 | −20.1158 | −23.7402 | −20.5360 | −20.3522 | −5.7084 | −39.2822 | −26.2212 | 8.5049 |
| | 0.25 | 184.8758 | −15.2361 | −20.5330 | −20.3014 | −23.9261 | −20.7160 | −20.3228 | −5.8898 | −39.4587 | −26.3944 | 8.3025 |
| 1.0 | 0.01 | 88.4806 | −7.1668 | −9.8128 | −9.6662 | −11.4623 | −9.8938 | −10.1645 | −2.4828 | −19.2812 | −12.7427 | 4.5925 |
| | 0.02 | 88.6414 | −7.1844 | −9.8304 | −9.6847 | −11.4809 | −9.9118 | −10.1615 | −2.5010 | −19.2988 | −12.7600 | 4.5722 |
| | 0.03 | 88.8022 | −7.2020 | −9.8481 | −9.7033 | −11.4995 | −9.9298 | −10.1586 | −2.5191 | −19.3165 | −12.7773 | 4.5520 |
| | 0.04 | 88.9630 | −7.2196 | −9.8658 | −9.7218 | −11.5181 | −9.9478 | −10.1556 | −2.5373 | −19.3341 | −12.7946 | 4.5317 |
| | 0.05 | 89.1238 | −7.2371 | −9.8835 | −9.7404 | −11.5367 | −9.9658 | −10.1527 | −2.5554 | −19.3518 | −12.8119 | 4.5115 |
| | 0.10 | 89.9278 | −7.3251 | −9.9718 | −9.8332 | −11.6296 | −10.0558 | −10.1380 | −2.6462 | −19.4400 | −12.8985 | 4.4103 |
| | 0.15 | 90.7319 | −7.4130 | −10.0602 | −9.9260 | −11.7226 | −10.1457 | −10.1232 | −2.7369 | −19.5283 | −12.9851 | 4.3090 |
| | 0.20 | 91.5359 | −7.5009 | −10.1486 | −10.0187 | −11.8155 | −10.2357 | −10.1085 | −2.8277 | −19.6165 | −13.0716 | 4.2078 |
| | 0.25 | 92.3399 | −7.5888 | −10.2369 | −10.1115 | −11.9084 | −10.3257 | −10.0938 | −2.9184 | −19.7048 | −13.1582 | 4.1066 |
| 5.0 | 0.01 | 17.5394 | −1.3865 | −1.9152 | −1.8706 | −2.2051 | −1.9271 | −1.9247 | −0.4541 | −3.8169 | −2.4861 | 0.8470 |
| | 0.02 | 17.5715 | −1.3901 | −1.9188 | −1.8743 | −2.2088 | −1.9307 | −1.9241 | −0.4578 | −3.8205 | −2.4896 | 0.8430 |
| | 0.03 | 17.6037 | −1.3936 | −1.9223 | −1.8780 | −2.2125 | −1.9343 | −1.9235 | −0.4614 | −3.8240 | −2.4931 | 0.8389 |
| | 0.04 | 17.6359 | −1.3971 | −1.9258 | −1.8817 | −2.2162 | −1.9379 | −1.9230 | −0.4650 | −3.8275 | −2.4965 | 0.8349 |
| | 0.05 | 17.6680 | −1.4006 | −1.9294 | −1.8854 | −2.2199 | −1.9415 | −1.9224 | −0.4686 | −3.8310 | −2.5000 | 0.8308 |
| | 0.10 | 17.8288 | −1.4182 | −1.9471 | −1.9040 | −2.2385 | −1.9595 | −1.9194 | −0.4868 | −3.8487 | −2.5173 | 0.8106 |
| | 0.15 | 17.9896 | −1.4358 | −1.9647 | −1.9225 | −2.2571 | −1.9775 | −1.9165 | −0.5049 | −3.8663 | −2.5346 | 0.7904 |
| | 0.20 | 18.1505 | −1.4534 | −1.9824 | −1.9411 | −2.2757 | −1.9955 | −1.9135 | −0.5231 | −3.8840 | −2.5519 | 0.7701 |
| | 0.25 | 18.3113 | −1.4709 | −2.0001 | −1.9596 | −2.2943 | −2.0135 | −1.9106 | −0.5412 | −3.9016 | −2.5693 | 0.7499 |
| 50.0 | 0.01 | 1.5776 | −0.0860 | −0.1383 | −0.1165 | −0.1222 | −0.1346 | −0.0708 | 0.0023 | −0.3375 | −0.1784 | 0.0043 |
| | 0.02 | 1.5808 | −0.0863 | −0.1386 | −0.1169 | −0.1226 | −0.1350 | −0.0707 | 0.0020 | −0.3378 | −0.1788 | 0.0039 |
| | 0.03 | 1.5841 | −0.0867 | −0.1390 | −0.1173 | −0.1229 | −0.1353 | −0.0707 | 0.0016 | −0.3382 | −0.1791 | 0.0035 |
| | 0.04 | 1.5873 | −0.0870 | −0.1394 | −0.1177 | −0.1233 | −0.1357 | −0.0706 | 0.0012 | −0.3385 | −0.1795 | 0.0031 |
| | 0.05 | 1.5905 | −0.0874 | −0.1397 | −0.1180 | −0.1237 | −0.1360 | −0.0705 | 0.0009 | −0.3389 | −0.1798 | 0.0027 |
| | 0.10 | 1.6066 | −0.0892 | −0.1415 | −0.1199 | −0.1255 | −0.1378 | −0.0702 | −0.0009 | −0.3406 | −0.1815 | 0.0007 |
| | 0.15 | 1.6226 | −0.0909 | −0.1432 | −0.1217 | −0.1274 | −0.1396 | −0.0700 | −0.0028 | −0.3424 | −0.1833 | −0.0014 |
| | 0.20 | 1.6387 | −0.0927 | −0.1450 | −0.1236 | −0.1292 | −0.1414 | −0.0697 | −0.0046 | −0.3442 | −0.1850 | −0.0034 |
| | 0.25 | 1.6548 | −0.0944 | −0.1468 | −0.1254 | −0.1311 | −0.1432 | −0.0694 | −0.0064 | −0.3459 | −0.1867 | −0.0054 |

The covariance matrix $\sigma_m$ for the ten FA is obtained as

$$\sigma_m = \begin{pmatrix} 0.071432 & 0.002637 & -0.00135 & -0.02882 & 0.005344 & -0.08015 & -0.00532 & -0.00882 & -0.06686 & 0.077392 \\ 0.002637 & 0.003526 & -0.00068 & -0.00885 & -0.00231 & 0.009269 & -0.01361 & 0.010385 & -0.00028 & -0.00988 \\ -0.00135 & -0.00068 & 0.005772 & 0.022111 & -0.00124 & 0.027262 & -0.00928 & -0.00906 & 0.004978 & -0.03258 \\ -0.02882 & -0.00885 & 0.022111 & 0.14254 & -0.00234 & 0.145541 & -0.03258 & -0.06107 & 0.041408 & -0.16313 \\ 0.005344 & -0.00231 & -0.00124 & -0.00234 & 0.006315 & -0.0502 & 0.011895 & -0.01906 & -0.00733 & 0.052826 \\ -0.08015 & 0.009269 & 0.027262 & 0.145541 & -0.0502 & 1.063518 & -0.11513 & 0.094139 & 0.116084 & -1.11225 \\ -0.00532 & -0.01361 & -0.00928 & -0.03258 & 0.011895 & -0.11513 & 0.290171 & -0.2112 & -0.01386 & 0.127339 \\ -0.00882 & 0.010385 & -0.00906 & -0.06107 & -0.01906 & 0.094139 & -0.2112 & 0.345718 & -0.00523 & -0.10046 \\ -0.06686 & -0.00028 & 0.004978 & 0.041408 & -0.00733 & 0.116084 & -0.01386 & -0.00523 & 0.071741 & -0.11861 \\ 0.077392 & -0.00988 & -0.03258 & -0.16313 & 0.052826 & -1.11225 & 0.127339 & -0.10046 & -0.11861 & 1.172205 \end{pmatrix}. \quad (86)$$

**Table 4**
The investor's optimal portfolio returns from the FA for varying $\gamma$ and $r$, and $z^* = 0.6$.

| $\gamma$ | $r$ | FA1 | FA2 | FA3 | FA4 | FA5 | FA6 | FA7 | FA8 | FA9 | FA10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.1 | 0.01 | 268.8250 | 285.0189 | 289.6520 | 297.9750 | 289.4218 | 80.6414 | 252.1436 | 337.9064 | 297.6860 | 254.9872 |
| | 0.02 | 268.4339 | 284.6306 | 289.2754 | 297.6010 | 289.0307 | 80.0999 | 251.7479 | 337.5179 | 297.2949 | 254.5388 |
| | 0.03 | 268.0428 | 284.2422 | 288.8988 | 297.2270 | 288.6395 | 79.5583 | 251.3522 | 337.1295 | 296.9037 | 254.0905 |
| | 0.04 | 267.6517 | 283.8539 | 288.5222 | 296.8530 | 288.2484 | 79.0168 | 250.9565 | 336.7410 | 296.5126 | 253.6422 |
| | 0.05 | 267.2605 | 283.4655 | 288.1456 | 296.4790 | 287.8572 | 78.4753 | 250.5609 | 336.3525 | 296.1214 | 253.1939 |
| | 0.10 | 265.3050 | 281.5238 | 286.2625 | 294.6090 | 285.9014 | 75.7676 | 248.5824 | 334.4103 | 294.1657 | 250.9523 |
| | 0.15 | 263.3494 | 279.5821 | 284.3795 | 292.7391 | 283.9457 | 73.0600 | 246.6040 | 332.4680 | 292.2100 | 248.7106 |
| | 0.20 | 261.3939 | 277.6403 | 282.4964 | 290.8691 | 281.9899 | 70.3523 | 244.6255 | 330.5257 | 290.2543 | 246.4690 |
| | 0.25 | 259.4383 | 275.6986 | 280.6133 | 288.9991 | 280.0341 | 67.6446 | 242.6471 | 328.5835 | 288.2985 | 244.2274 |
| 0.5 | 0.01 | 53.8275 | 57.0652 | 57.9827 | 59.6422 | 57.9440 | 16.2808 | 50.4937 | 67.6338 | 59.5950 | 51.1048 |
| | 0.02 | 53.7493 | 56.9875 | 57.9074 | 59.5674 | 57.8658 | 16.1725 | 50.4146 | 67.5561 | 59.5168 | 51.0152 |
| | 0.03 | 53.6711 | 56.9098 | 57.8321 | 59.4926 | 57.7875 | 16.0642 | 50.3355 | 67.4784 | 59.4386 | 50.9255 |
| | 0.04 | 53.5929 | 56.8322 | 57.7567 | 59.4178 | 57.7093 | 15.9559 | 50.2563 | 67.4007 | 59.3603 | 50.8359 |
| | 0.05 | 53.5146 | 56.7545 | 57.6814 | 59.3430 | 57.6311 | 15.8476 | 50.1772 | 67.3230 | 59.2821 | 50.7462 |
| | 0.10 | 53.1235 | 56.3662 | 57.3048 | 58.9690 | 57.2399 | 15.3061 | 49.7815 | 66.9346 | 58.8910 | 50.2979 |
| | 0.15 | 52.7324 | 55.9778 | 56.9282 | 58.5950 | 56.8488 | 14.7645 | 49.3858 | 66.5461 | 58.4998 | 49.8495 |
| | 0.20 | 52.3413 | 55.5895 | 56.5516 | 58.2210 | 56.4576 | 14.2230 | 48.9901 | 66.1577 | 58.1087 | 49.4012 |
| | 0.25 | 51.9502 | 55.2011 | 56.1750 | 57.8470 | 56.0665 | 13.6815 | 48.5944 | 65.7692 | 57.7175 | 48.9529 |
| 1.0 | 0.01 | 26.9528 | 28.5710 | 29.0240 | 29.8506 | 29.0093 | 8.2358 | 25.2875 | 33.8497 | 29.8336 | 25.6196 |
| | 0.02 | 26.9137 | 28.5321 | 28.9864 | 29.8132 | 28.9702 | 8.1816 | 25.2479 | 33.8109 | 29.7945 | 25.5747 |
| | 0.03 | 26.8746 | 28.4933 | 28.9487 | 29.7758 | 28.9310 | 8.1274 | 25.2084 | 33.7720 | 29.7554 | 25.5299 |
| | 0.04 | 26.8355 | 28.4545 | 28.9110 | 29.7384 | 28.8919 | 8.0733 | 25.1688 | 33.7332 | 29.7163 | 25.4851 |
| | 0.05 | 26.7964 | 28.4156 | 28.8734 | 29.7010 | 28.8528 | 8.0191 | 25.1292 | 33.6943 | 29.6772 | 25.4402 |
| | 0.10 | 26.6008 | 28.2215 | 28.6851 | 29.5140 | 28.6572 | 7.7484 | 24.9314 | 33.5001 | 29.4816 | 25.2161 |
| | 0.15 | 26.4053 | 28.0273 | 28.4968 | 29.3270 | 28.4616 | 7.4776 | 24.7335 | 33.3059 | 29.2860 | 24.9919 |
| | 0.20 | 26.2097 | 27.8331 | 28.3085 | 29.1400 | 28.2661 | 7.2068 | 24.5357 | 33.1117 | 29.0905 | 24.7677 |
| | 0.25 | 26.0142 | 27.6389 | 28.1202 | 28.9530 | 28.0705 | 6.9361 | 24.3379 | 32.9174 | 28.8949 | 24.5436 |
| 5.0 | 0.01 | 5.4531 | 5.7756 | 5.8571 | 6.0173 | 5.8615 | 1.7997 | 5.1225 | 6.8225 | 6.0245 | 5.2313 |
| | 0.02 | 5.4453 | 5.7678 | 5.8496 | 6.0098 | 5.8537 | 1.7889 | 5.1146 | 6.8147 | 6.0167 | 5.2224 |
| | 0.03 | 5.4374 | 5.7601 | 5.8420 | 6.0023 | 5.8458 | 1.7780 | 5.1067 | 6.8069 | 6.0089 | 5.2134 |
| | 0.04 | 5.4296 | 5.7523 | 5.8345 | 5.9948 | 5.8380 | 1.7672 | 5.0988 | 6.7992 | 6.0011 | 5.2044 |
| | 0.05 | 5.4218 | 5.7445 | 5.8270 | 5.9874 | 5.8302 | 1.7564 | 5.0909 | 6.7914 | 5.9933 | 5.1955 |
| | 0.10 | 5.3827 | 5.7057 | 5.7893 | 5.9500 | 5.7911 | 1.7022 | 5.0513 | 6.7526 | 5.9541 | 5.1506 |
| | 0.15 | 5.3436 | 5.6669 | 5.7516 | 5.9126 | 5.7520 | 1.6481 | 5.0117 | 6.7137 | 5.9150 | 5.1058 |
| | 0.20 | 5.3045 | 5.6280 | 5.7140 | 5.8752 | 5.7128 | 1.5939 | 4.9722 | 6.6749 | 5.8759 | 5.0610 |
| | 0.25 | 5.2654 | 5.5892 | 5.6763 | 5.8378 | 5.6737 | 1.5398 | 4.9326 | 6.6360 | 5.8368 | 5.0161 |
| 50.0 | 0.01 | 0.6156 | 0.6466 | 0.6445 | 0.6548 | 0.6532 | 0.3516 | 0.5854 | 0.7413 | 0.6675 | 0.6440 |
| | 0.02 | 0.6149 | 0.6459 | 0.6438 | 0.6540 | 0.6524 | 0.3505 | 0.5846 | 0.7406 | 0.6667 | 0.6431 |
| | 0.03 | 0.6141 | 0.6451 | 0.6430 | 0.6533 | 0.6517 | 0.3494 | 0.5838 | 0.7398 | 0.6659 | 0.6422 |
| | 0.04 | 0.6133 | 0.6443 | 0.6423 | 0.6526 | 0.6509 | 0.3483 | 0.5830 | 0.7390 | 0.6652 | 0.6413 |
| | 0.05 | 0.6125 | 0.6435 | 0.6415 | 0.6518 | 0.6501 | 0.3473 | 0.5822 | 0.7382 | 0.6644 | 0.6404 |
| | 0.10 | 0.6086 | 0.6396 | 0.6378 | 0.6481 | 0.6462 | 0.3418 | 0.5783 | 0.7344 | 0.6605 | 0.6359 |
| | 0.15 | 0.6047 | 0.6358 | 0.6340 | 0.6443 | 0.6423 | 0.3364 | 0.5743 | 0.7305 | 0.6565 | 0.6314 |
| | 0.20 | 0.6008 | 0.6319 | 0.6302 | 0.6406 | 0.6384 | 0.3310 | 0.5704 | 0.7266 | 0.6526 | 0.6269 |
| | 0.25 | 0.5969 | 0.6280 | 0.6265 | 0.6368 | 0.6345 | 0.3256 | 0.5664 | 0.7227 | 0.6487 | 0.6225 |

Note that $\sigma_{10} = \sigma_{S_2} = \sigma_{P_2}$ and $\sigma_m = \sigma_{S_3} = \sigma_{P_3}$. The inflation volatility matrix is given by $\sigma_{P1} = \sigma_{S1} = 0.17\mathbf{1}$ and income growth rate volatility matrix is given by $\sigma_{P4} = \sigma_{S4} = 0.25\mathbf{1}$. The volatilities $\sigma_{P2}$ with respect to stocks and $\sigma_{S3}$ with respect to FA are obtained from Table 1. The market price of risks $\theta_i(t) = \theta_{i0} + \theta_{i1}t, i = I, S, P, \beta$ for inflation, stocks, FAs and income are given as follows respectively:

$$\begin{aligned}
\theta_{I0} &= [0.5920, 0, 0, 0, 0, 0, 0, 0, 0, 0]' \\
\theta_{S0} &= [0.004, 0.0041, 0.0042, 0.0034, 0.0039, 0.0044, 0.0039, 0.004, 0.0032, 0.0026]' \\
\theta_{P0} &= [0.0034, 0.0035, 0.0042, 0.0052, 0.002, 0.0011, 0.0023, 0.0025, 0.0031, 0.0013]' \\
\theta_{\beta 0} &= [0.0017, 0.0022, 0.0019, 0.005, 0.0092, 0.0023, 0, 0, 0, 0]' \\
\mathrm{SP} &= \theta_{I1} = 0.00022\mathbf{e},\ \theta_{S1} = 0.0001\mathbf{e},\ \theta_{P1} = 0.0002\mathbf{e},\ \theta_{\beta 1} = 0.00001\mathbf{e}.
\end{aligned}$$

**Table 5**
The investor's optimal portfolio returns from the FA for varying $g_S$ and $g_F$, and $z^* = 0.6$.

| $g_S$ | $g_F$ | $\gamma$ | FA1 | FA2 | FA3 | FA4 | FA5 | FA6 | FA7 | FA8 | FA9 | FA10 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.1 | 1.9711 | 0.1 | 228.7446 | 242.4706 | 246.4143 | 253.4859 | 246.2169 | 69.1310 | 214.6014 | 287.3310 | 253.2232 | 216.9673 |
| | | 0.2 | 114.4055 | 121.2678 | 123.2349 | 126.7679 | 123.1400 | 34.6464 | 107.3352 | 143.6933 | 126.6422 | 108.5406 |
| | | 0.4 | 57.2359 | 60.6664 | 61.6451 | 63.4090 | 61.6016 | 17.4040 | 53.7020 | 71.8745 | 63.3518 | 54.3272 |
| | | 0.6 | 38.1793 | 40.4660 | 41.1152 | 42.2893 | 41.0888 | 11.6566 | 35.8243 | 47.9349 | 42.2549 | 36.2561 |
| | | 0.8 | 28.6511 | 30.3658 | 30.8503 | 31.7295 | 30.8324 | 8.7828 | 26.8855 | 35.9651 | 31.7065 | 27.2205 |
| | | 1.0 | 22.9341 | 24.3056 | 24.6913 | 25.3936 | 24.6786 | 7.0586 | 21.5222 | 28.7832 | 25.3775 | 21.7992 |
| | | 2.0 | 11.5002 | 12.1853 | 12.3734 | 12.7218 | 12.3709 | 3.6101 | 10.7955 | 14.4194 | 12.7194 | 10.9565 |
| | | 4.0 | 5.7832 | 6.1252 | 6.2144 | 6.3859 | 6.2170 | 1.8859 | 5.4322 | 7.2376 | 6.3903 | 5.5352 |
| | | 6.0 | 3.8776 | 4.1052 | 4.1614 | 4.2739 | 4.1658 | 1.3112 | 3.6444 | 4.8436 | 4.2806 | 3.7281 |
| | | 8.0 | 2.9247 | 3.0951 | 3.1349 | 3.2179 | 3.1401 | 1.0238 | 2.7506 | 3.6466 | 3.2258 | 2.8245 |
| | | 10.0 | 2.3530 | 2.4891 | 2.5190 | 2.5843 | 2.5247 | 0.8514 | 2.2142 | 2.9284 | 2.5929 | 2.2824 |
| | | 50.0 | 0.5236 | 0.5499 | 0.5481 | 0.5569 | 0.5555 | 0.2996 | 0.4980 | 0.6302 | 0.5676 | 0.5475 |
| | | 100.0 | 0.2949 | 0.3075 | 0.3018 | 0.3034 | 0.3093 | 0.2306 | 0.2834 | 0.3429 | 0.3144 | 0.3307 |
| | | 150.0 | 0.2187 | 0.2267 | 0.2197 | 0.2189 | 0.2273 | 0.2076 | 0.2119 | 0.2472 | 0.2300 | 0.2584 |
| 0.2 | 1.8711 | 0.1 | 240.9695 | 255.4289 | 259.5835 | 267.0330 | 259.3755 | 72.8256 | 226.0704 | 302.6869 | 266.7563 | 228.5627 |
| | | 0.2 | 120.5196 | 127.7487 | 129.8209 | 133.5428 | 129.7210 | 36.4980 | 113.0715 | 151.3728 | 133.4104 | 114.3413 |
| | | 0.4 | 60.2947 | 63.9086 | 64.9396 | 66.7977 | 64.8938 | 18.3341 | 56.5720 | 75.7157 | 66.7375 | 57.2306 |
| | | 0.6 | 40.2197 | 42.6286 | 43.3126 | 44.5494 | 43.2847 | 12.2795 | 37.7389 | 50.4967 | 44.5132 | 38.1937 |
| | | 0.8 | 30.1823 | 31.9886 | 32.4990 | 33.4252 | 32.4802 | 9.2522 | 28.3223 | 37.8872 | 33.4010 | 28.6753 |
| | | 1.0 | 24.1598 | 25.6046 | 26.0109 | 26.7507 | 25.9975 | 7.4358 | 22.6724 | 30.3215 | 26.7337 | 22.9642 |
| | | 2.0 | 12.1148 | 12.8366 | 13.0346 | 13.4017 | 13.0320 | 3.8031 | 11.3725 | 15.1901 | 13.3991 | 11.5421 |
| | | 4.0 | 6.0923 | 6.4526 | 6.5465 | 6.7272 | 6.5493 | 1.9867 | 5.7225 | 7.6243 | 6.7318 | 5.8310 |
| | | 6.0 | 4.0848 | 4.3246 | 4.3838 | 4.5023 | 4.3884 | 1.3812 | 3.8392 | 5.1024 | 4.5094 | 3.9273 |
| | | 8.0 | 3.0810 | 3.2606 | 3.3024 | 3.3899 | 3.3079 | 1.0785 | 2.8976 | 3.8415 | 3.3982 | 2.9754 |
| | | 10.0 | 2.4788 | 2.6221 | 2.6536 | 2.7225 | 2.6597 | 0.8969 | 2.3326 | 3.0849 | 2.7315 | 2.4043 |
| | | 50.0 | 0.5516 | 0.5793 | 0.5774 | 0.5866 | 0.5852 | 0.3156 | 0.5246 | 0.6639 | 0.5979 | 0.5768 |
| | | 100.0 | 0.3107 | 0.3239 | 0.3179 | 0.3196 | 0.3259 | 0.2430 | 0.2986 | 0.3613 | 0.3312 | 0.3484 |
| | | 150.0 | 0.2304 | 0.2388 | 0.2314 | 0.2306 | 0.2394 | 0.2187 | 0.2233 | 0.2604 | 0.2423 | 0.2722 |
| 0.4 | 1.6711 | 0.1 | 269.8083 | 285.9982 | 290.6500 | 298.9910 | 290.4171 | 81.5413 | 253.1261 | 338.9119 | 298.6812 | 255.9167 |
| | | 0.2 | 134.9432 | 143.0375 | 145.3577 | 149.5250 | 145.2458 | 40.8660 | 126.6037 | 169.4888 | 149.3767 | 128.0255 |
| | | 0.4 | 67.5107 | 71.5571 | 72.7115 | 74.7920 | 72.6602 | 20.5283 | 63.3425 | 84.7772 | 74.7245 | 64.0799 |
| | | 0.6 | 45.0332 | 47.7303 | 48.4961 | 49.8810 | 48.4650 | 13.7491 | 42.2554 | 56.5400 | 49.8404 | 42.7647 |
| | | 0.8 | 33.7944 | 35.8169 | 36.3884 | 37.4255 | 36.3674 | 10.3595 | 31.7119 | 42.4214 | 37.3984 | 32.1071 |
| | | 1.0 | 27.0512 | 28.6689 | 29.1238 | 29.9522 | 29.1088 | 8.3257 | 25.3858 | 33.9503 | 29.9332 | 25.7125 |
| | | 2.0 | 13.5647 | 14.3728 | 14.5946 | 15.0056 | 14.5917 | 4.2582 | 12.7335 | 17.0080 | 15.0027 | 12.9234 |
| | | 4.0 | 6.8214 | 7.2248 | 7.3300 | 7.5323 | 7.3331 | 2.2244 | 6.4074 | 8.5368 | 7.5375 | 6.5288 |
| | | 6.0 | 4.5737 | 4.8421 | 4.9084 | 5.0412 | 4.9136 | 1.5465 | 4.2987 | 5.7131 | 5.0491 | 4.3973 |
| | | 8.0 | 3.4498 | 3.6508 | 3.6977 | 3.7956 | 3.7038 | 1.2076 | 3.2443 | 4.3012 | 3.8049 | 3.3315 |
| | | 10.0 | 2.7755 | 2.9360 | 2.9712 | 3.0483 | 2.9780 | 1.0042 | 2.6117 | 3.4541 | 3.0584 | 2.6921 |
| | | 50.0 | 0.6176 | 0.6486 | 0.6465 | 0.6568 | 0.6552 | 0.3534 | 0.5874 | 0.7434 | 0.6695 | 0.6458 |
| | | 100.0 | 0.3479 | 0.3627 | 0.3559 | 0.3579 | 0.3649 | 0.2720 | 0.3343 | 0.4045 | 0.3709 | 0.3900 |
| | | 150.0 | 0.2580 | 0.2674 | 0.2591 | 0.2583 | 0.2681 | 0.2449 | 0.2500 | 0.2916 | 0.2713 | 0.3048 |

The volatilities vectors of the models and values of other parameters are given as following:

$$
\begin{aligned}
\sigma_{1I} &= [0.17, 0, 0, 0, 0, 0, 0, 0, 0, 0],\\
\sigma_{2I} &= \begin{matrix}[0.00793201, 0.00640514, 0.00690392, 0.00894483, 0.00611450,\\ 0.00125935, 0.00723558, 0.00581460, 0.00672061, 0.00551647]\end{matrix},\\
\sigma_{3I} &= \begin{matrix}[0.08294039, 0.07906477, 0.06014989, 0.01572066, 0.0744897,\\ 0.06331878, 0.02590696, 0.02622017, 0.08168191, 0.06328203]\end{matrix},\\
\sigma_{4I} &= [0.11, 0.21, 0.19, 0.15, 0.17, 0.23, 0, 0, 0, 0],\\
\sigma_{1\beta} &= [0.17, 0, 0, 0, 0, 0, 0, 0, 0, 0],\\
\sigma_{2\beta} &= \begin{matrix}[0.000793201, 0.00640514, 0.00690392, 0.00894483, 0.00611450,\\ 0.00125935, 0.00723558, 0.00581460, 0.00672061, 0.00551647]\end{matrix},\\
\sigma_{3\beta} &= \begin{matrix}[0.08294039, 0.07906477, 0.06014989, 0.01572066, 0.0744897,\\ 0.06331878, 0.02590696, 0.02622017, 0.08168191, 0.06328203]\end{matrix},\\
\sigma_{4\beta} &= [0.15, 0.124, 0.25, 0.172, 0.189, 0.21, 0.19, 0.10, 0, 0],\\
\xi &= 0.12 + 0.007t,\ \beta_0 = 0.002,\ \epsilon = 0.0002,\ K_0 = 0.1,\ \eta = 0.005,\\
\bar{r} &= 0.024 + 0.001t,\ r = 0.05 + 0.001t.
\end{aligned}
$$

Again, we take $t \in [0, 10]$. The collateral security ratio is given by $K_\xi(t) = k_0 e^{0.12t+0.007\frac{t^2}{2}}$, where $K_\xi(t) = \frac{K(t)}{X(t)}$, $k_0 = \frac{K(0)}{X(0)} = 0.1, \beta_0 = 0.002, h_\beta = 0.2, \epsilon = 0.0002, \eta(\omega) = 0.005, g_F = 1 + z^* + g_K - g_S$.

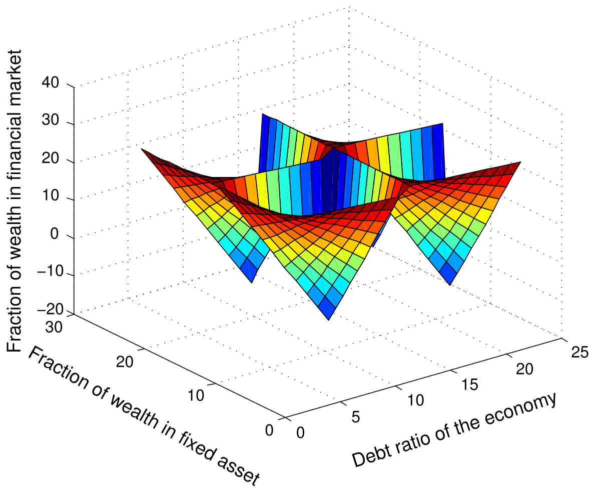


**Fig. 1.** A slice plot showing the fraction of wealth in stocks for a given fraction of wealth in FA and debt ratio of an investor for $g_S = 0.4, \gamma = 0.1$.

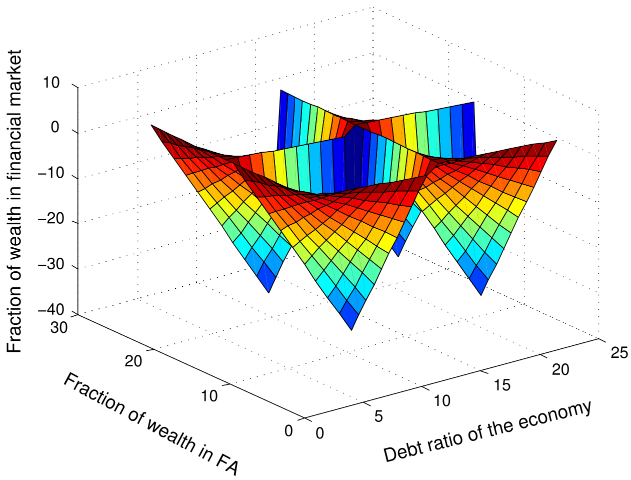


**Fig. 2.** A slice plot showing the fraction of wealth in stocks for a given fraction of wealth in FA and debt ratio of an investor for $\gamma = 0.5, g_S = 0.4, \pi_F = 120$.

Figs. 1 and 2 show the fraction of wealth in stocks for a given fraction of wealth in FA and debt ratio of an investor. We also observe that as the debt profile of an investor increases the portfolio values of an investor in stocks decreases and vice versa. It implies that high debt ratio of an economy can crash the FM. Figs. 3 and 4 show the impact of debt and portfolio value in FAs on portfolio value in stocks. It is observed that as the portfolio value in FAs increases the portfolio value in stocks increases and vice versa. It is also observed that debt ratio of an investor has negative impact on the portfolio values over time. Fig. 5 shows the fraction of wealth in the FM against an investor debt profile over time. One will see clearly that the higher the debt, the lower the portfolio value in the FM. Hence, an investor FM will face financial crisis if the debt profile continuing to increase over time. This shows that for the FM to boom, an investor must try as much as possible to minimize her debt profile over time.

Table 2 shows the optimal portfolio returns from the fixed assets for varying $z^*$ and $\gamma$, and by setting $r = 0.02$. It is observed that as $\gamma$ increases the portfolio value in FAs decreases, and vice versa. Again, it is observed that as $z^*$ increases the portfolio value in FAs decreases as well, and vice versa. This shows that high debt is not good for an investor.

Table 3 shows the investor's optimal portfolio returns from the FMs for varying value of $\gamma$ and $r$, and by setting $z^* = 0.6$. It is observed from Table 3 that as interest rate increases, the portfolio value in stocks decreases drastically and the portfolio value in cash account increases drastically. This implies that high interest rate can have negative impact on the investment portfolio in stocks and have positive impact on cash account. It therefore follows that when the interest rate is high all the investment in stocks should remain in cash account over time. It is observed that as the value of $\gamma$ increases, the portfolio value in stocks increases and the portfolio value in cash account decreases and vice versa.

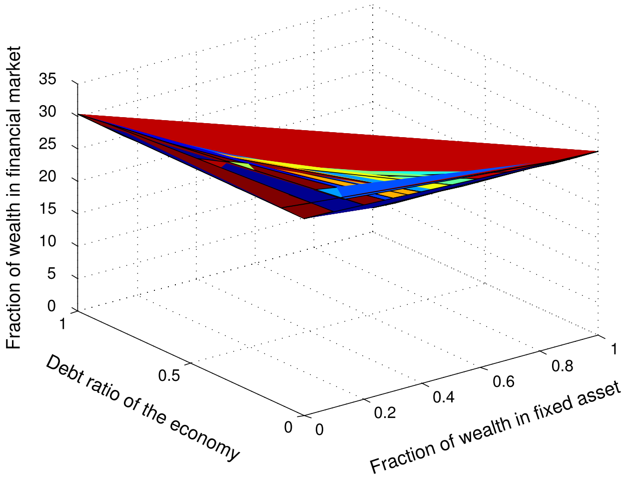


**Fig. 3.** The portfolio value of an investor in stock market versus portfolio value in FA and debt ratio of an investor for $g_S = 0.4, \gamma = 0.1$.

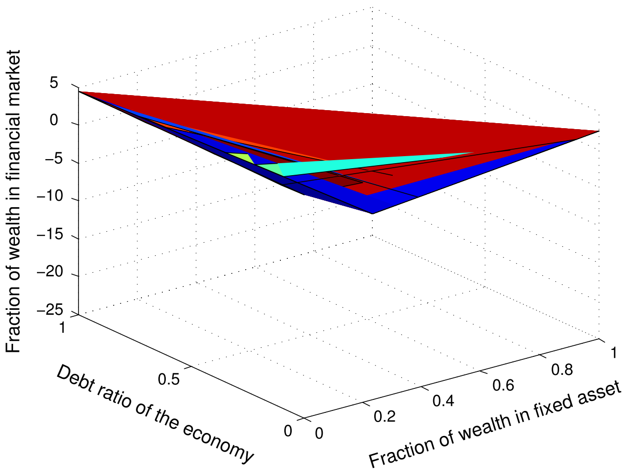


**Fig. 4.** The portfolio value of an investor in stock market versus portfolio value in FA and debt ratio of an investor for $g_S = 0.4, \gamma = 0.9$.

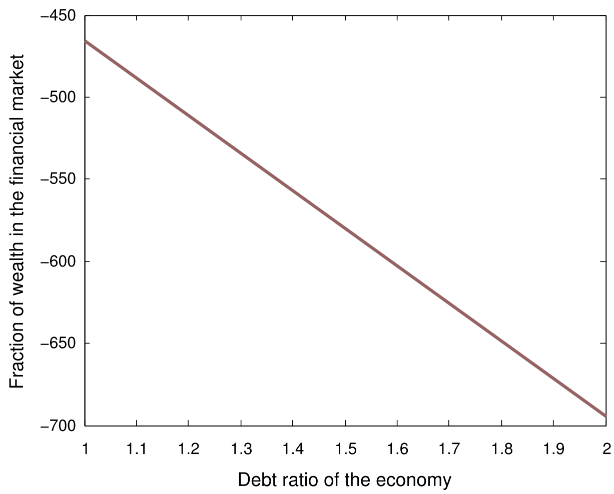


**Fig. 5.** Fraction of wealth in the FM versus debt profile of an investor for $\gamma = 0.5, g_S = 0.4, \pi_F = 120$.

Table 4 shows the investor's optimal portfolio returns from the FA for varying value of $\gamma$ and $r$, and by setting $z^* = 0.6$. It is observed that as the interest rate increases, the portfolio values of the FAs decreases and vice versa. This implies that high interest rate will have adverse effect on the investment portfolio of an investor. It is also observed in Table 4 that the higher the value of $\gamma$ will reduce the investment portfolio of an investor. We can see that high interest rate will discourage the investment in both stocks and fixed assets, and encourage investment in cash account over time. We can also see that high value of $\gamma$, for all other parameters remain fixed, will encourage investment in stocks and discourage investment in both cash account and FAs. Hence, for all other parameters remain fixed, for high interest rate, investment in cash account should be encouraged, and for high value of $\gamma$, investment in stocks should be encouraged.

Table 5 shows the investor's optimal portfolio returns from the FA for varying $g_S$ and $g_F$, and by setting $z^* = 0.6$. Interestingly, it is observed that as the financial market is booming, the proportion that goes to FAs will reduce but the portfolio value of an investor in FAs will continue to increase over time. It implies that FM has strong influence on the investment in FAs, but reverse is not the case.

## 9. Conclusion

Here, we give the concluding remarks of the paper. This paper presented a theoretical and an empirical study of an optimal investment management strategy and debt profile of an investor in a financial crisis. The investor's investment faces four background risks: inflation, investment, fixed asset and income risks. A backup security on the investor's liability was considered. Real wealth of an investor was determined. The investment strategies and consumption plan of an investor that are exposed to diffusion and credit risks were analyzed. The market was divided into two: financial and fixed asset market. The underlying assets in financial market considered are stocks and a riskless asset. The optimal investment strategies in the financial and fixed asset markets are obtained. The optimal consumption and debt ratio of an investor are obtained. It was found that investment in fixed assets can hedge credit risks in the stock market investment portfolio. Furthermore, the investment portfolio in fixed assets was found to depend inversely on the optimal debt ratio of an investor. It was also found that the investment portfolio in fixed assets depends directly on the investment in financial market, but the reverse is not the case.

Empirically, we found that

1. as the debt profile of an investor increases, the portfolio values of an investor in stocks decreases and vice versa. It implies that high debt ratio of an economy can crash the FM.
2. as the portfolio value in stocks increases, the portfolio value in FAs increases as well.
3. debt ratio of an investor has negative impact on the investment portfolio in the FA over time.
4. as interest rate increases, the portfolio value in stocks and FAs decreases and the portfolio value in cash account increases over time, and vice versa.
5. the increase in the value of $\gamma$, for all other parameters remain fixed, will encourage investment in stocks and discourage investment in both cash account and FAs.
6. as the financial market is booming, the proportion that goes to FAs will reduce, but the portfolio value of an investor in FAs will continue to increase over time.

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