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Maximizing and minimizing investment concentration with constraints of budget and investment risk

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Highlight for reviewers

Thank you for your comments. We answer the questions:

(1) On distribution of return rate
(2) On phase transition of this investment system
(3) Comparison with numerical result and the finding derived using our proposed method.
(4) On the interpretation of chi

Best regards

Maximizing and minimizing investment concentration with constraints of budget and investment risk

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Abstract

In this paper, as a first step in examining the properties of a feasible portfolio subset that is characterized by budget and risk constraints, we assess the maximum and minimum of the investment concentration using replica analysis. To do this, we apply an analytical approach of statistical mechanics. We note that the optimization problem considered in this paper is the dual problem of the portfolio optimization problem discussed in the literature, and we verify that these optimal solutions are also dual. We also present numerical experiments, in which we use the method of steepest descent that is based on Lagrange's method of undetermined multipliers, and we compare the numerical results to those obtained by replica analysis in order to assess the effectiveness of our proposed approach.

Keywords: Portfolio optimization, Replica analysis, Investment risk, Investment concentration

1. Introduction

The portfolio optimization problem is one of the most important research topics in the area of mathematical finance, and it is well known that the investment risk can be reduced by diversifying assets in accordance with the knowledge obtained from the optimal solutions to this problem [1, 2]. The pioneering re-

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search on this topic was reported by Markowitz in 1952 [3, 4], and it is still an active area of research [5, 6]. Several recent studies have considered investment models that use the analytical approaches developed in cross-disciplinary fields, such as replica analysis, belief propagation methods, and using the distribution of the eigenvalues of random matrices [7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. For instance, Ciliberti *et al* [7, 8] used replica analysis in the limit of absolute zero temperature to examine the minimal investment risk per asset when using the absolute deviation model or the expected shortfall model. Kondor *et al* [9] quantified the sensitivity to noise for several risk functions, including the in-sample risk, the out-sample risk, and the predicted risk. Moreover, Pafka *et al* [10] investigated the relationship between the number of investment periods and the value of assets, as well as various investment risks such as the predicted risk and the practical risk. Shinzato [11] used replica analysis to show that for the mean-variance model, the minimal investment risk and its concentration are self-averaging. Furthermore, Shinzato [12] developed the replica approach used in [11] so as to analyze the mean-variance model with the nonidentical variances of asset returns. Moreover, Shinzato *et al* [13] developed an algorithm based on a belief propagation method to solve for the optimal portfolio when using the mean-variance model and the absolute deviation model, and they proved the Konno-Yamazaki conjecture for a quenched disordered system. Varga-Haszonits *et al* [14] used replica analysis to investigate the minimal variance of deviation of difference between actual return and expected return and the efficient frontier for the mean-variance model under budget and return constraints. In addition, Shinzato [15] examined the minimizing investment risk problem under the constraints of budget and expected return using the duality of portfolio optimization problem. Recently, Shinzato [16] used replica analysis to investigate the minimal investment risk for the mean-variance model with budget and investment concentration constraints.

Of the studies discussed above, the minimal investment risk for a mean-variance model with a number of constraints was analyzed only in Refs. [15, 16] as a natural extension of the mean-variance model with a budget constraint considered in Ref. [11]; it turns out that the dual problem is implied in these portfolio optimization problems. In order to better understand these optimization problems, we use the dual structure to analyze them. However, in the various investigations of this problem that have used analytical approaches that were developed in cross-disciplinary fields (including replica analysis and an approach based on using the distribution of the eigenvalues of random matrices), there are a few studies that analyze the potential of an investment system that proactively employs a dual structure and the dual problem [15, 16]. Shinzato [15] first applied the dual structure of portfolio optimization problem so as to analyze the minimization of investment risk under the constraints of budget and expected return and the maximization of expected return under the constraints of budget and investment risk from multilateral viewpoints. Though the minimization of investment risk under budget constraint and investment concentration is already reported in [16], we need to supportedly examine the dual problem of already-discussed optimization. Since the investment concentration is a statistics similar to Herfindahl-Hirschman index (HHI), investigating the range (or the upper and lower bounds) of investment concentration of optimal portfolio is valid when investing optimally. The relationship between investment concentration and HHI is already shown in [15]. As a first step in discussing the mathematical framework of a dual structure, our aim in this paper is to solve the dual problem of the portfolio optimization problem [16] and to clarify the dual structure of these optimization problems.

This paper is organized as follows: in section 2, we state the dual problem of the portfolio optimization problem with budget and investment concentration constraints, as discussed in Ref. [16]. In section 3, we use replica analysis to investigate this dual problem. In section 4, we compare the results of the replica analysis to those estimated by numerical experiments and evaluate the effectiveness of our proposed method. In section 5, we present our conclusions and discuss areas of future work.

2. Model Setting

As in Refs. [11, 12, 13, 15, 16], we consider a stable investment market in which there is no regulation of short selling and in which there are N assets. A portfolio of asset $i(=1,\cdots,N)$ is notated as w_i, and a portfolio of N assets is notated as $\vec{w}=(w_1,w_2,\cdots,w_N)^{\mathrm{T}}\in\mathbf{R}^N$. We will use the notation T to mean the transpose of a vector or a matrix. For simplicity, we assume that short selling is not regulated, and we note that w_i is not always nonnegative. We assume p scenarios (or priods), and the return rate of asset i in scenario $\mu(=1,\cdots,p)$ is $\bar{x}_{i\mu}$, where the return rates are independently distributed with a mean $E_X[\bar{x}_{i\mu}]$ and unit variance. We will consider the feasible portfolio subset $W(\kappa)$, which is subject to the following constraints on the budget and risk constraint:

$$N = \sum_{i=1}^{N} w_i, \tag{1}$$

$$N\kappa\varepsilon = \frac{1}{2}\sum_{\mu=1}^{p}\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N} w_i\left(\bar{x}_{i\mu}-E_X[\bar{x}_{i\mu}]\right)\right)^2, \tag{2}$$

where equation (1) is the budget constraint used in Refs. [11, 12, 13, 15, 16], equation (2) is a risk constraint, and ε is the minimal investment risk $\varepsilon=\frac{\alpha-1}{2}$. Note that equation (2) implies that the investment risk for N assets is $\kappa(\geq 1)$ times the minimal investment risk $N\varepsilon$. We will call κ the risk coefficient, and the scenario ratio is defined as $\alpha=p/N$. In addition, the modified return rate $x_{i\mu}$ is defined as $x_{i\mu}=\bar{x}_{i\mu}-E_X[\bar{x}_{i\mu}]$, and so the feasible portfolio subset $W(\kappa)\subseteq\mathbf{R}^N$ can be rewritten as follows:

$$W(\kappa) = \left\{\vec{w}\in\mathbf{R}^N \,\middle|\, N=\vec{w}^{\mathrm{T}}\vec{e}, N\kappa\varepsilon=\frac{1}{2}\sum_{\mu=1}^{p}\left(\frac{\vec{w}^{\mathrm{T}}\vec{x}_\mu}{\sqrt{N}}\right)^2\right\}, \tag{3}$$

where the unit vector $\vec{e}=(1,\cdots,1)^{\mathrm{T}}\in\mathbf{R}^N$, and the modified return rate vector is $\vec{x}_\mu=(x_{1\mu},x_{2\mu},\cdots,x_{N\mu})^{\mathrm{T}}\in\mathbf{R}^N$. That is, the Wishart matrix $XX^{\mathrm{T}}\in\mathbf{R}^{N\times N}$ defined by the modified return rate matrix $X=\left\{\frac{x_{i\mu}}{\sqrt{N}}\right\}\in\mathbf{R}^{N\times p}$ is the metric of the Mahalanobis distance (or, more accurately, half the squared Mahalanobis distance), $\frac{1}{2}\vec{w}^{\mathrm{T}}XX^{\mathrm{T}}\vec{w}$, which is constant. We need to examine the portfolios included in the feasible subset $W(\kappa)$ in order to investigate the properties of the investment market. We will use the following statistic, which has been used previously in the literature (e.g., [11]):

$$q_w = \frac{1}{N}\sum_{i=1}^{N} w_i^2. \quad (4)$$

For instance, when $\kappa = 1$, the optimal solution is unique; when $\kappa > 1$, the feasible subset $W(\kappa)$ is not empty, and if we can determine the range investment concentrations, then we can determine the number of portfolios in that subset.

Finally, we note that a previous study [11] examined the optimal solution that minimizes the investment risk in equation (2) under the budget constraint in equation (1), and it also analyzed the minimal investment risk. A different study [16] examined the optimal solution that minimizes the investment risk in equation (2) under the budget constraint in equation (1) and the investment concentration constraint in equation (4), and again, it analyzed the minimal investment risk. We note that this study [16], which discusses the portfolio optimization problem with two constraints, is a natural extension of the previous study [11], which considered only a single constraint. In this paper, we interchange the investment concentration constraint and the object function (the investment risk) to consider the dual of the problem considered in Ref. [16].

3. Replica analysis

In this section, we use replica analysis [17, 18] to investigate the optimization problem discussed above. The Hamiltonian in this investment system is

$$\mathcal{H}(\vec{w}) = \frac{1}{2}\sum_{i=1}^{N} w_i^2. \quad (5)$$

Following the approach of statistical mechanics, the partition function $Z(\kappa, X)$ of the inverse temperature β is

$$Z(\kappa, X) = \int_{-\infty}^{\infty} d\vec{w} P(\vec{w}|\kappa, X) e^{\beta\mathcal{H}(\vec{w})}, \quad (6)$$

$$P(\vec{w}|\kappa, X) = \delta\left(\sum_{i=1}^{N} w_i - N\right)\delta\left(N\kappa\varepsilon - \frac{1}{2}\sum_{\mu=1}^{p}\left(\frac{\vec{w}^{\mathrm{T}}\vec{x}_\mu}{\sqrt{N}}\right)^2\right), \quad (7)$$

where $X = \left\{ \frac{x_{i\mu}}{\sqrt{N}} \right\} \in \mathbf{R}^{N\times p}$ is the return rate matrix. From this, the maximum and minimum of the investment concentration, $q_{w,\max}$ and $q_{w,\min}$, respectively, can be derived using the following formula:

$$\begin{aligned} q_{w,\max} &= \max_{\vec{w}\in W(\kappa)} \left\{ \frac{1}{N} \sum_{i=1}^{N} w_i^2 \right\} \\ &= \lim_{\beta\to\infty} \frac{2}{N} \frac{\partial}{\partial\beta} \log Z(\kappa, X), \end{aligned} \tag{8}$$

$$\begin{aligned} q_{w,\min} &= \min_{\vec{w}\in W(\kappa)} \left\{ \frac{1}{N} \sum_{i=1}^{N} w_i^2 \right\} \\ &= \lim_{\beta\to-\infty} \frac{2}{N} \frac{\partial}{\partial\beta} \log Z(\kappa, X). \end{aligned} \tag{9}$$

In order to assess the bounds of the investment concentration, we use the unified viewpoint approach of statistical mechanics, although we do not use the Boltzmann factor, which is widely used in the literature of statistical mechanics. Since this representation maintains the mathematical structure of this model, we can analyze both bounds within large limits of the inverse temperature β. In addition, in order to examine the typical behavior of this investment system, we need to evaluate the typical maximum and minimum investment concentrations. That is, we must rigorously average the right-hand side in equation (8) and equation (9) over the return rate of assets.

In a way similar to that used in previous studies [11, 16], we used replica analysis and the ansatz of the replica symmetry solution (see Appendix Appendix A for details), as follows:

$$\begin{aligned} \phi &= \lim_{N\to\infty} \frac{1}{N} E_X[\log Z(\kappa, X)] \\ &= \mathop{\mathrm{Extr}}_{k,\theta,\chi_w,q_w,\tilde{\chi}_w,\tilde{q}_w} \left\{ -k + \kappa\theta\varepsilon + \frac{1}{2}(\chi_w + q_w)(\tilde{\chi}_w - \tilde{q}_w) \right. \\ &\quad + \frac{q_w\tilde{q}_w}{2} + \frac{\beta}{2}(\chi_w + q_w) + \frac{k^2}{2\tilde{\chi}_w} - \frac{1}{2}\log\tilde{\chi}_w + \frac{\tilde{q}_w}{2\tilde{\chi}_w} \\ &\quad \left. - \frac{\alpha}{2}\log(1 + \theta\chi_w) - \frac{\alpha\theta q_w}{2(1 + \theta\chi_w)} \right\}, \end{aligned} \tag{10}$$

where $\mathrm{Extr}_m f(m)$ is the extremum of function $f(m)$ with respect to m , and the replica symmetry solution is evaluated at $a, b = 1, 2, \cdots, n$, as follows:

$$q_{wab} = \begin{cases} \chi_w + q_w & a = b \\ q_w & a \neq b \end{cases}, \tag{11}$$

$$\tilde{q}_{wab} = \begin{cases} \tilde{\chi}_w - \tilde{q}_w & a = b \\ -\tilde{q}_w & a \neq b \end{cases}, \tag{12}$$

$$k_a = k, \tag{13}$$

$$\theta_a = \theta, \tag{14}$$

where k is the auxiliary variable with respect to equation (1), and θ is the auxiliary variable with respect to equation (2). From this, the extremum conditions
130 in equation (10) are derived as follows:

$$k = \tilde{\chi}_w, \tag{15}$$

$$\chi_w = \frac{1}{\tilde{\chi}_w}, \tag{16}$$

$$q_w = 1 + \frac{\tilde{q}_w}{\tilde{\chi}_w^2}, \tag{17}$$

$$\tilde{\chi}_w + \beta = \frac{\alpha\theta}{1 + \theta\chi_w}, \tag{18}$$

$$\tilde{q}_w = \frac{\alpha\theta^2 q_w}{(1 + \theta\chi_w)^2}, \tag{19}$$

$$\frac{\kappa(\alpha - 1)}{2} = \frac{\alpha\chi_w}{2(1 + \theta\chi_w)} + \frac{\alpha q_w}{2(1 + \theta\chi_w)^2}. \tag{20}$$

In order to obtain the maximum and minimum, we need to take the limit as $|\beta| \to \infty$; we use the results presented in Refs. [11, 16]. Then, we assume $\theta\chi_w \sim O(1)$ and $\frac{\beta}{\theta} \sim O(1)$, and so we obtain

$$\theta\chi_w = \frac{1 \pm \sqrt{\alpha - \frac{\alpha}{\kappa}}}{\alpha - 1}, \tag{21}$$

$$\frac{\beta}{\theta} = \pm\frac{(\alpha - 1)^2\sqrt{\alpha - \frac{\alpha}{\kappa}}}{2\alpha - \frac{\alpha}{\kappa} \pm (\alpha + 1)\sqrt{\alpha - \frac{\alpha}{\kappa}}}. \tag{22}$$

From these, we then obtain

$$\chi_w = \pm\frac{(\alpha - 1)\sqrt{\alpha - \frac{\alpha}{\kappa}}}{\beta\left(\alpha \pm \sqrt{\alpha - \frac{\alpha}{\kappa}}\right)}, \tag{23}$$

$$q_w = \frac{\left(\sqrt{\alpha\kappa} \pm \sqrt{\kappa - 1}\right)^2}{\alpha - 1}, \tag{24}$$

where, from the seventh term in equation (10), we have $-\frac{1}{2}\log\tilde{\chi}_w = \frac{1}{2}\log\chi_w$, since $\chi_w > 0$. Note that if $\beta > 0$, the χ_w, and q_w are both positive, and if $\beta < 0$, they are both negative. Moreover, from Eqs. (8), (9), and (10), we obtain

$$\begin{aligned}\lim_{N\to\infty} 2\frac{\partial}{\partial\beta}\left\{\frac{1}{N}E_X[\log Z(\kappa, X)]\right\} &= 2\frac{\partial\phi}{\partial\beta} \\ &= \chi_w + q_w, \end{aligned} \tag{25}$$

is obtained. Since χ_w is close to 0, then when $|\beta| \to \infty$, we obtain

$$q_{w,\max} = \frac{\left(\sqrt{\alpha\kappa} + \sqrt{\kappa - 1}\right)^2}{\alpha - 1}, \tag{26}$$

$$q_{w,\min} = \frac{\left(\sqrt{\alpha\kappa} - \sqrt{\kappa - 1}\right)^2}{\alpha - 1}. \tag{27}$$

Four points should be noted here. First, both bounds of the investment concentration are consistent when $\kappa = 1$, and so $q_{w,\max} = q_{w,\min} = \frac{\alpha}{\alpha-1}$. Second, the maximum investment concentration $q_{w,\max}$ has no upper bound, while the minimum investment concentration $q_{w,\min}$ has a lower bound at $\kappa = \frac{\alpha}{\alpha-1}$, and so $q_{w,\min} = 1$. Third, the optimization problem discussed in the literature is the dual problem of the one considered in the present work. When $\tau = q_{w,\max}$, $\kappa = \frac{(\alpha+1)\tau - 1 - 2\sqrt{\alpha\tau(\tau-1)}}{\alpha-1}$, and so the investment risk per asset $\varepsilon' = \kappa\varepsilon$ is calculated as follows:

$$\varepsilon' = \frac{\alpha\tau + \tau - 1 - 2\sqrt{\alpha\tau(\tau - 1)}}{2}. \tag{28}$$

We note that this coincides with the minimal investment risk per asset obtained in our previous studies [16, 19]. That is, the portfolio in $W(\kappa)$ that maximizes the investment concentration corresponds to the portfolio in

$$R(\tau) = \left\{\vec{w} \in \mathbf{R}^N \left| \vec{w}^{\mathrm{T}}\vec{e} = N, \vec{w}^{\mathrm{T}}\vec{w} = N\tau \right.\right\} \tag{29}$$

that minimizes the investment risk. If $\tau = q_{w,\min}$ and $\kappa = \frac{(\alpha+1)\tau - 1 + 2\sqrt{\alpha\tau(\tau-1)}}{\alpha-1}$, then the investment risk per asset $\varepsilon'' = \kappa\varepsilon$ is

$$\varepsilon'' = \frac{\alpha\tau + \tau - 1 + 2\sqrt{\alpha\tau(\tau - 1)}}{2}; \tag{30}$$

this corresponds to the maximal investment risk per asset found in Refs. [16, 19]; that is, the portfolio in $W(\kappa)$ that minimizes the investment concentration corresponds to the portfolio in $R(\tau)$ in equation (29) that maximizes the investment risk.

The fourth point considers the annealed disordered system for this investing strategy (for a detailed explanation of annealed and quenched disordered systems, see [11]). From our previous studies [11, 13], the minimal expected investment risk per asset of an annealed disordered system is $\varepsilon^{\rm OR} = \frac{\alpha}{2}$, and so the risk constraint in equation (2) is replaced by

$$\begin{aligned} N\kappa\varepsilon^{\rm OR} &= \frac{1}{2}\sum_{\mu=1}^{p} E_X\left[\left(\frac{\vec{w}^{\rm T}\vec{x}_\mu}{\sqrt{N}}\right)^2\right] \\ &= \frac{\alpha}{2}\sum_{i=1}^{N} w_i^2, \end{aligned} \tag{31}$$

where $E_X[x_{i\mu}x_{j\mu}] = \delta_{ij}$. From this, the feasible portfolio subset of the annealed disordered system is calculated as follows:

$$W^{\rm OR}(\kappa) = \left\{\vec{w}\in\mathbf{R}^N \left| N = \vec{w}^{\rm T}\vec{e}, \frac{N\kappa\alpha}{2} = \frac{\alpha}{2}\sum_{i=1}^{N} w_i^2 \right.\right\}. \tag{32}$$

Thus, the maximum and minimum of the investment concentration $q_w^{\rm OR}$ are the same:

$$q_w^{\rm OR} = \kappa. \tag{33}$$

The feasible portfolio subset $W(\kappa)$ in equation (3) is determined by the portfolio $\vec{w}$ for which half of the squared Mahalanobis distance is consistent; note that the metric of the Mahalanobis distance is defined by the Wishart matrix $XX^{\rm T}$, which is derived from the return rate matrix $X = \left\{\frac{x_{i\mu}}{\sqrt{N}}\right\} \in \mathbf{R}^{N\times p}$. However, in general, since this feasible portfolio subset $W(\kappa)$ is not isotropic, the portfolio closest to the origin (which minimizes the investment concentration) and the one farthest from the origin (which maximizes the investment concentration) are uniquely determined. However, since the feasible portfolio subset of the annealed disordered system $W^{\mathrm{OR}}(\kappa)$ is isotropic, this implies that the maximum and minimum investment concertation are the same.

4. Numerical Experiments

In order to evaluate the effectiveness of our proposed approach, we numerically assess the maximum and minimum investment concentration, $q_{w,\max}$ and $q_{w,\min}$, respectively, and compare the results with those obtained by replica analysis. We replace the feasible portfolio subset $W(\kappa)$ in equation (3) with constraint conditions using Lagrange's method of undetermined multipliers, and the object function of Lagrange's method $L(\vec{w}, k, \theta)$ is defined as follows:

$$L(\vec{w}, k, \theta) = \frac{1}{2}\vec{w}^{\mathrm{T}}\vec{w} + k(N - \vec{e}^{\mathrm{T}}\vec{w}) + \theta\left(\frac{1}{2}\vec{w}^{\mathrm{T}}J\vec{w} - N\kappa\varepsilon\right), \tag{34}$$

where k, θ are the auxiliary variables, and the i,jth component of the Wishart matrix $J(= XX^{\mathrm{T}}) = \{J_{ij}\} \in \mathbf{R}^{N\times N}$ is

$$J_{ij} = \frac{1}{N}\sum_{\mu=1}^{p} x_{i\mu}x_{j\mu}. \tag{35}$$

It is necessary to evaluate the optimal solution of the object function of Lagrange's method, $L(\vec{w}, k, \theta)$, in order to determine the maximum and minimum of investment concentration. We used the following method of steepest descent:

$$\vec{w}^{s+1} = \vec{w}^{s} - \eta_w \frac{\partial L(\vec{w}, k, \theta)}{\partial \vec{w}}, \tag{36}$$

$$k^{s+1} = k^{s} + \eta_k \frac{\partial L(\vec{w}, k, \theta)}{\partial k}, \tag{37}$$

$$\theta^{s+1} = \theta^{s} + \eta_\theta \frac{\partial L(\vec{w}, k, \theta)}{\partial \theta}, \tag{38}$$

where, at step s, the portfolio is $\vec{w}^s = (w_1^s, w_2^s, \cdots, w_N^s)^{\mathrm{T}} \in \mathbf{R}^N$ and the auxiliary variables are $k^s, \theta^s \in \mathbf{R}$; also, $\vec{w}^0 = \vec{e}$ and $k^0 = \theta^0 = 1$. When $\eta_k, \eta_\theta, \eta_w > 0$, we can determine the minimum, and when $\eta_k, \eta_\theta, \eta_w < 0$, we can determine the maximum. The stopping condition is that $\Delta = \sum_{i=1}^{N} |w_i^s - w_i^{s+1}| + |k^s - k^{s+1}| + |\theta^s - \theta^{s+1}|$ is less than δ.

From this, we obtain the M return rate matrices, $X^1, \cdots, X^M$, where the mth return rate matrix is $X^m = \left\{\frac{x_{i\mu}^m}{\sqrt{N}}\right\} \in \mathbf{R}^{N\times p}$, with respect to the risk coefficient κ, using $q_{w,\max}(\kappa, X^m)$ and $q_{w,\min}(\kappa, X^m)$, as estimated using the algorithm given above. These are calculated as follows:

$$q_{w,\max}(\kappa) = \frac{1}{M}\sum_{m=1}^{M} q_{w,\max}(\kappa, X^m), \tag{39}$$

$$q_{w,\min}(\kappa) = \frac{1}{M}\sum_{m=1}^{M} q_{w,\min}(\kappa, X^m), \tag{40}$$

where the return rate of asset i, $x_{i\mu}^m$, is independently and identically distributed with zero mean and unit variance.

We performed numerical experiments with the following settings: $N = 1000, p = 3000$, $\alpha = p/N = 3$, and $M = 10$. When seeking the minimum, we used $\delta = 10^{-5}$, $\eta_k = 10^{-1}, \eta_\theta = 10^{-5}$, and $\eta_w = 10^{-1}$, and when seeking the maximum, we used $\eta_k = -10^{-1}, \eta_\theta = -10^{-5}$, and $\eta_w = -10^{-1}$. The results of the replica analysis and numerical experiments are shown in Figs. 1 and 2. The horizontal axis shows the investment concentration q_w, and the vertical axis shows the risk coefficient κ. Solid lines are the results of the replica analysis (Eqs. (26) and (27)) and the asterisks with error bars are the results of the numerical simulation (Eqs. (39) and (40)). The figures show that the results of the replica analysis are consistent with those of the numerical simulation, and so we can use replica analysis to accurately analyze the portfolio optimization problem.

5. Conclusion and Future work

In the present study, we used replica analysis, which was developed for cross-disciplinary research, to analyze the duality problem of the portfolio optimization problem with several constraint conditions, which has been considered in our previous studies [11, 12, 13, 14, 15, 16]. We determined a feasible portfolio that maximizes the investment concentration subject to budget and risk constraints, and one that minimizes the investment concentration. We applied a canonical ensemble analysis to a large, complicated system with respect to this optimization problem with several restrictions. From a unified viewpoint, we were able to derive the maximum and minimum investment concentrations from the subset of feasible portfolios. The portfolio optimization problem considered in this paper is the dual of the optimization problem discussed in our previous study [16], and we verified that the optimal solutions possess the duality structure. In the numerical experiments, we used the method of steepest descent that is based on Lagrange's method of undetermined multipliers, and we compared the numerical and theoretical results to verify our proposed approach.

In this and our previous studies [11, 16, 19], we analyzed a portfolio optimization problem subject to several constraints. In the future, we intend to further examine the complicated relationship between this and the dual problem in more general situations. In addition, we intend to examine the effects of regulating short selling.

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Appendix A. Replica analysis

In this appendix, we explain the replica analysis used in the present paper. As in Ref. [11], $E_X[Z^n(\kappa, X)]$, $n \in \mathbf{Z}$, is defined as follows:

$$\begin{aligned} & E_X[Z^n(\kappa, X)] \\ = & \frac{1}{(2\pi)^{\frac{Nn}{2}+pn}} \int_{-\infty}^{\infty} \prod_{a=1}^{n} d\vec{w}_a d\vec{u}_a d\vec{v}_a E_X \left[\exp\left(\frac{\beta}{2} \sum_{i=1}^{N} \sum_{a=1}^{n} w_{ia}^2 \right.\right. \\ & + \sum_{a=1}^{n} k_a \left(\sum_{i=1}^{N} w_{ia} - N \right) + \sum_{a=1}^{n} \theta_a \left(N\kappa\varepsilon - \frac{1}{2} \sum_{\mu=1}^{p} v_{\mu a}^2 \right) \\ & \left.\left. + i \sum_{\mu=1}^{p} \sum_{a=1}^{n} u_{\mu a} \left(v_{\mu a} - \frac{1}{\sqrt{N}} \sum_{i=1}^{N} x_{i\mu} w_{ia} \right) \right) \right]. \end{aligned} \quad (A.1)$$

245 We caluculate the configuration average at first,

$$\begin{aligned} & E_X \left[\exp\left(-\frac{i}{\sqrt{N}} \sum_{i=1}^{N} \sum_{\mu=1}^{p} x_{i\mu} \sum_{a=1}^{n} u_{\mu a} w_{ia} \right) \right] \\ = & \mathop{\mathrm{Extr}}_{Q_w, \tilde{Q}_w} \exp\left(-\frac{1}{2} \sum_{\mu=1}^{p} \sum_{a,b} u_{\mu a} u_{\mu b} q_{wab} - \frac{1}{2} \sum_{a,b} \tilde{q}_{wab} \left(\sum_{i=1}^{N} w_{ia} w_{ib} - N q_{wab} \right) \right), \end{aligned} \quad (A.2)$$

where the notation $\sum_{a,b}$ means $\sum_{a=1}^{n} \sum_{b=1}^{n}$ and $Q_w = \{q_{wab}\} \in \mathbf{R}^{n\times n}$, $\tilde{Q}_w = \{\tilde{q}_{wab}\} \in \mathbf{R}^{n\times n}$ and we have the order parameters

$$q_{wab} = \frac{1}{N} \sum_{i=1}^{N} w_{ia} w_{ib}, \quad (A.3)$$

and the conjugate parameters $\tilde{q}_{wab}$. Moreover,

$$\begin{aligned} & \frac{1}{(2\pi)^{\frac{Nn}{2}}} \int_{-\infty}^{\infty} \prod_{i=1}^{N} \prod_{a=1}^{n} dw_{ia} \exp\left(\sum_{i=1}^{N} \left\{ \frac{\beta}{2} \sum_{a=1}^{n} w_{ia}^2 - \frac{1}{2} \sum_{a,b} \tilde{q}_{wab} w_{ia} w_{ib} + \sum_{a=1}^{n} k_a w_{ia} \right\} \right) \\ = & \exp\left(-\frac{N}{2} \log\det\left| \tilde{Q}_w \right| + \frac{N\beta}{2} \mathrm{Tr} Q_w + \frac{N}{2} \vec{k}^{\mathrm{T}} \tilde{Q}_w^{-1} \vec{k} \right), \end{aligned} \quad (A.4)$$

is also assessed. In addition,

$$\frac{1}{(2\pi)^{pn}}\int_{-\infty}^{\infty}\prod_{\mu=1}^{p}\prod_{a=1}^{n}du_{\mu a}dv_{\mu a}\exp\left(\sum_{\mu=1}^{p}\left\{-\frac{1}{2}\sum_{a,b}q_{wab}u_{\mu a}u_{\mu b}-\frac{1}{2}\sum_{a=1}^{n}\theta_a v_{\mu a}^2 +i\sum_{a=1}^{n}u_{\mu a}v_{\mu a}\right\}\right)$$
$$= \exp\left(-\frac{p}{2}\log\det\begin{vmatrix} Q_w & -iI \\ -iI & \Theta\end{vmatrix}\right), \tag{A.5}$$

is evaluated. Here, k_a is the auxiliary variable with respect to equation (1), and θ_a is the auxiliary variable with respect to equation (2). In addition, $\vec{k} = (k_1, \cdots, k_n)^{\mathrm{T}} \in \mathbf{R}^n$, $\vec{\theta} = (\theta_1, \cdots, \theta_n)^{\mathrm{T}} \in \mathbf{R}^n$, $\vec{e} = (1, \cdots, 1)^{\mathrm{T}} \in \mathbf{R}^n$, and $\Theta = \mathrm{diag}\{\theta_1, \theta_2, \cdots, \theta_n\} \in \mathbf{R}^{n\times n}$.

In the thermodynamic limit of the number of assets N, we obtain

$$\lim_{N\to\infty}\frac{1}{N}\log E_X[Z^n(\kappa, X)]$$
$$= \frac{\beta}{2}\mathrm{Tr}Q_w - \vec{k}^{\mathrm{T}}\vec{e} + \kappa\varepsilon\vec{\theta}^{\mathrm{T}}\vec{e} + \frac{1}{2}\mathrm{Tr}Q_w\tilde{Q}_w + \frac{1}{2}\vec{k}^{\mathrm{T}}\tilde{Q}_w^{-1}\vec{k} -\frac{1}{2}\log\det|\tilde{Q}_w| - \frac{\alpha}{2}\log\det\begin{vmatrix} Q_w & -iI \\ -iI & \Theta\end{vmatrix}. \tag{A.6}$$

If we substitute the replica symmetry solutions from Eqs. (11) to (14) into equation (A.6), we obtain

$$\lim_{N\to\infty}\frac{1}{N}\log E_X[Z^n(\kappa, X)]$$
$$= \frac{n\beta}{2}(\chi_w + q_w) - nk + n\kappa\varepsilon\theta + \frac{n}{2}(\chi_w + q_w)(\tilde{\chi}_w - \tilde{q}_w) - \frac{n(n-1)}{2}q_w\tilde{q}_w + \frac{nk^2}{2(\tilde{\chi}_w - n\tilde{q}_w)} - \frac{n-1}{2}\log\tilde{\chi}_w - \frac{1}{2}\log(\tilde{\chi}_w - n\tilde{q}_w) - \frac{\alpha(n-1)}{2}\log(1+\theta\chi_w) - \frac{\alpha}{2}\log(1+\theta\chi_w + n\theta q_w). \tag{A.7}$$

From this, equation (10) is also obtained.

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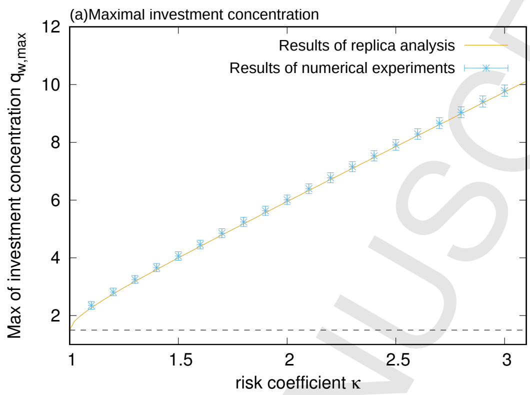

Figure 1: Comparison of the maximal investment concentration obtained by the replica analysis to that obtained in the numerical experiments; $\alpha = p/N = 3$. The horizontal axis shows the risk coefficient κ, and the vertical axis shows the minimal investment concentration $q_{w,\max}$. The solid line (orange) shows the results of the replica analysis, the asterisks with error bars (blue) show the results of the numerical simulation, and the dashed line shows the investment concentration at $\kappa = 1$, that is, $\frac{\alpha}{\alpha-1}$.

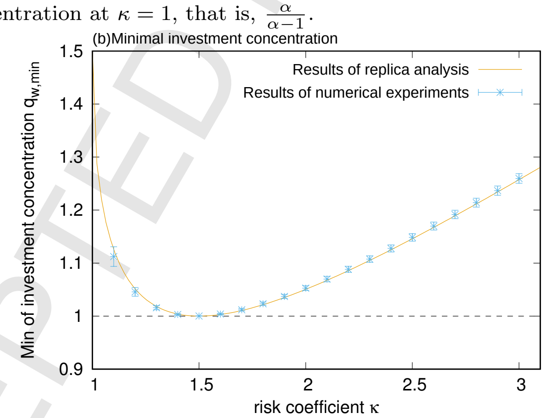

Figure 2: Comparison of the minimal investment concentration obtained by the replica analysis to that obtained in the numerical experiments; $\alpha = p/N = 3$. The horizontal axis shows the risk coefficient κ, and the vertical axis shows the maximal investment concentration $q_{w,\min}$. The solid line (orange) shows the results of the replica analysis, the asterisks with error bars (blue) show the results of the numerical simulation, and the dashed line shows the minimal investment concentration, that is, unity.