Signal Covariance Matrix Optimization for Transmit Beamforming in MIMO Radars

Tuomas Aittomäki and Visa Koivunen
Signal Processing Laboratory, SMARAD CoE
Helsinki University of Technology
P.O. Box 3000, FI-02015 HUT, Finland
Email: taittoma@wooster.hut.fi, visa@wooster.hut.fi

***Abstract*—MIMO radars use multiple waveforms simultaneously to improve performance. A beamforming method that exploits this waveform diversity has been proposed previously. This method works by optimizing the covariance matrix of the waveforms to obtain an approximation of a desired beampattern. The previous method uses gradient descent to optimize the beampattern with the constraint on the power of each antenna element. We show how this method can be extended to obtain rank-deficient covariance matrices and also to handle the total power constraint. The conjugate gradient method is used in addition to the gradient descent.**

In this paper, we also propose converting the constrained beampattern optimization problem into an unconstrained one. This can be done by using the method of Lagrange multipliers, but also removing all constraints and then scaling the result so that the total power constraint is satisfied. Using this approach, the beampattern optimization can be written as a least squares problem.

I. Introduction

MIMO radar is a novel radar paradigm in which radars use multiple waveforms simultaneously to improve performance. The development of MIMO radars has been motivated by the advances made in the MIMO communication systems, which have been shown to increase rate and reliability of radio links.

Recently, several techniques extending the MIMO concept to radars have been proposed[1]–[6]. These techniques aim at enhancing the radar, but they differ how the waveforms are employed. The waveform diversity can be used to improve the performance of the radars in many different ways.

A beamforming method using the correlation of the transmitted waveforms was introduced in [1]. Beamforming is useful in phased-array radars, which need to maximize the energy transmitted to the direction of a target. The beamforming method proposed in [1] allows much higher control over the beampattern than ordinary beamforming. For example, this method can be applied to do beamspoiling, which is useful in reducing backscatter clutter and scanning time[2]. The ability to form a wide focus is useful especially when transmitting fully correlated waveforms would result in a beam that is too narrow. Transmit beamforming using signal correlation offers flexibility in the beampattern synthesis that is not achieved with the traditional phased array techniques or MIMO radars that use completely uncorrelated signals.

This work was supported by the Finnish Defence Forces Technical Research Centre and Academy of Finland, Center of Excellence program

The beampattern synthesis can be done by changing the correlation matrix of the transmitted signals so that a cost function measuring the difference between the desired and the actual beampattern is minimized. An optimization method that results in full-rank covariance matrix was proposed in [1]. However, it can be useful to form a correlation matrix that is not of full rank if the number of available waveforms is smaller than the number of transmitters, for example. A method for obtaining a rank-deficient covariance matrix was previously discussed in [7], but in the suggested method, the problem needs to be first solved without the rank constraint and then a separate algorithm is applied to get the final result. We demonstrate that the rank of the correlation matrix can be easily limited simply by adjusting the method in [1].

A gradient descent method was used in [1] with a rotational update to prevent violation of constraints set on the power of the individual transmitters. In this paper, we show how the algorithm can be extended to constrain the total power of the antenna array. The use of different optimization algorithms in the beampattern optimization problem is also studied. In addition, we propose converting the constrained optimization problem into an unconstrained one. This can be done with the method of Lagrange multipliers. We also show that for total power constraint, it is possible to remove the constraint altogether and scale the result after optimization. Based on this property, we propose obtaining approximate beampatterns by formulating the beampattern optimization as a least squares problem.

This paper is organized as follows: The transmit beampattern synthesis method originally presented in [1] is briefly reviewed in Section II. We also show how to modify the method to limit the rank of the covariance matrix. The unconstrained version of the method is discussed in Section III and the least squares formulation in Section IV. Section V shows numerical results of the discussed optimization methods. Final conclusions are made in Section VI.

II. Beampattern Optimization

Beamforming can be done by modifying the covariance matrix of the transmitted signals. If the typical far-field and narrowband assumptions hold, the radiation intensity (power

978-1-4244-2110-7/08/$25.00 ©2007 IEEE

per solid angle) of a linear array of N elements is

$$I(\theta,\phi) = \frac{1}{4\pi}\mathbf{v}^H(\theta,\phi)\mathbf{R}\mathbf{v}(\theta,\phi), \tag{1}$$

where $\mathbf{v}(\theta)$ is an $N \times 1$ steering vector and $\mathbf{R}$ is the $N \times N$ covariance matrix of the signals transmitted by the array elements[1]. The steering vector for a linear array is defined as

$$\mathbf{v}(\theta) = \left[\exp(-j2\pi \tfrac{z_1}{\lambda}\sin\theta) \quad \ldots \quad \exp(-j2\pi \tfrac{z_N}{\lambda}\sin\theta)\right]^T, \tag{2}$$

where z_i's are the positions of the elements relative to a reference point in the array. The problem is finding signals with such $\mathbf{R}$ that the beampattern $I(\theta,\phi)$ would have a desired shape. Because the steering vector of a linear array depends only on θ, the same is true for the beampattern I.

In standard beamforming, each element of the array transmits one signal phase-shifted so that the phases are exactly the same in a desired direction. Assuming that all the elements can transmit with unit power, $\mathbf{R} = \mathbf{v}(\theta_0)\mathbf{v}^H(\theta_0)$, where θ_0 is the direction of interest, and thus,

$$I(\theta) = \frac{1}{4\pi}|\mathbf{v}^H(\theta_0)\mathbf{v}(\theta)|^2. \tag{3}$$

In the desired direction, the transmitted power is

$$I(\theta_0) = \frac{1}{4\pi}\|\mathbf{v}(\theta_0)\|^4 = \frac{N^2}{4\pi}, \tag{4}$$

so there is N-fold beamforming gain in power compared to a single transmitter. On the other hand, if a MIMO radar transmitting uncorrelated signals is used, $\mathbf{R} = \mathbf{I}$ and $I(\theta) = \frac{N}{4\pi}$ for any θ.

Traditional beamforming results in highly focused beampatterns, whereas the beampattern of a MIMO radar with uncorrelated signals would be omnidirectional. Sometimes it might be necessary to synthesize a beampattern that is between these two extremes so that wide focus areas can be formed without wasting power to directions that are of no interest. This can be achieved by adjusting the covariance matrix of the transmitted signals.

In order to obtain a beampattern of a desired shape, an appropriate covariance matrix $\mathbf{R}$ has to be found. This can be done by minimizing the difference between the desired and the actual beampatterns according to some error criterion that depends on $\mathbf{R}$. However, $\mathbf{R}$ cannot be chosen freely because the covariance matrix has to be positive-semidefinite. One way to ensure positive-semidefiniteness is to use the square root of $\mathbf{R}$ so that

$$\mathbf{R} = \mathbf{Q}\mathbf{Q}^H, \tag{5}$$

where $\mathbf{Q}$ is the square root[1]. Matrix $\mathbf{Q}$ was chosen to be a complex-valued $N \times N$ matrix in [1]. However, the maximum rank of $\mathbf{R}$ can be easily controlled with the size of $\mathbf{Q}$, as the rank of $\mathbf{R}$ can be at most the same as the number of columns in $\mathbf{Q}$.

Power of the transmitter elements imposes another constraint on $\mathbf{R}$. If all antenna elements are assumed to transmit at the same maximum power, all diagonal elements of $\mathbf{R}$ must be equal. As before, we assume each transmitter transmits with unit power so that the total power of the array would be N.

Suppose that $\mathbf{Q}$ consists of $1 \times M$ *row vectors* $\mathbf{q}_n$. The power constraint then requires that $\|\mathbf{q}_n\|^2 = 1$. In other words, the optimization of $\mathbf{Q}$ needs to be done on the *oblique manifold* of $\mathbb{C}^{N\times M}$[8].

The beampattern can be optimized by choosing $\mathbf{Q}$ to minimize a cost function C that measures the difference between the desired and the actual beampattern. The optimization method proposed in [1] is based gradient descent. The gradient of a cost function C with respect to matrix $\mathbf{Q}$ is projected first into the tangent space of the manifold. Denoting a row of the gradient matrix $\mathbf{G}$ by $\mathbf{g}_n$, the projection to the tangent space is

$$\mathbf{t}_n = \mathbf{g}_n - \frac{\mathbf{q}_n\mathbf{q}_n^H\mathbf{g}_n}{\|\mathbf{q}_n\|^2}. \tag{6}$$

A line search is then done on the manifold to minimize the cost function by rotating the vectors $\mathbf{q}_n$ by

$$\mathbf{q}_n^{(i+1)} = \mathbf{q}_n^{(i)}\cos(\alpha\|\mathbf{t}_n^{(i)}\|) + \frac{\mathbf{t}_n^{(i)}}{\|\mathbf{t}_n^{(i)}\|}\sin(\alpha\|\mathbf{t}_n^{(i)}\|), \tag{7}$$

where α is a scalar that is same for every n.

The optimization on the manifold can be alternatively done by retraction[8], so that

$$\mathbf{q}_n^{(i+1)} = \frac{\mathbf{q}_n^{(i)} + \beta\mathbf{t}_n^{(i)}}{\sqrt{\|\mathbf{q}_n^{(i)}\|^2 + \beta^2\|\mathbf{t}_n^{(i)}\|^2}}. \tag{8}$$

This methods uses a square root instead of the sine and cosine required in the rotational update. In addition to simple gradient descent, it is also possible to use the conjugate gradient method as was done in [9] for a different problem.

Instead of constraining the power of the individual transmitters, the total power can be required to be constant[7]. The total power is

$$\text{trace}(\mathbf{R}) = \text{trace}(\mathbf{Q}\mathbf{Q}^H) = \text{vec}^H(\mathbf{Q})\text{vec}(\mathbf{Q}), \tag{9}$$

so the problem is finding the optimal vector with a fixed 2-norm in $\mathbb{C}^{MN\times 1}$. Both the rotation and retraction can be easily modified to cope with this constraint.

As mentioned before, the maximum rank of $\mathbf{R}$ can be controlled with the number of columns in $\mathbf{Q}$. However, rank-one case poses a problem as the projection of the gradient in (6) is always zero for complex scalars. We propose solving this by optimizing on the oblique manifold in $\mathbb{R}^{N\times 2M}$. With the previous notation, this can be done simply by replacing $\mathbf{q}_n^H\mathbf{g}_n$ in (6) with

$$\text{real}(\mathbf{q}_n^T)\text{real}(\mathbf{g}_n) + \text{imag}(\mathbf{q}_n^T)\text{imag}(\mathbf{g}_n)$$

When M is greater than one, the results are very similar regardless of the space used.

III. Unconstrained Optimization

The beampattern optimization problem at hand can be written as

$$\min C(\mathbf{Q}) \qquad \text{s.t.} \qquad \|\mathbf{q}_n\|^2 = 1,\ n = 1\ldots N \tag{10}$$

We propose transforming this constrained optimization problem into unconstrained one. This can be done in several different ways, but we will consider the method of Lagrange multipliers in this paper.

The Lagrangian of the problem is

$$\mathcal{L}(\mathbf{Q}, \mathbf{m}) = C(\mathbf{Q}) + \mathbf{m}^T \mathbf{h}(Q) \tag{11}$$

where elements of the vector-valued constraint function $\mathbf{h}$ are

$$h_n(\mathbf{Q}) = \|\mathbf{q}_n\|^2 - 1 \tag{12}$$

in case of an elemental power constraint. For a total power constraint, the constraint equation is

$$h(\mathbf{Q}) = \sum_{n=1}^{N} \|\mathbf{q}_n\|^2 - N. \tag{13}$$

A critical point of the Lagrangian can be found numerically using, for example, a quasi-Newton method.

For certain cost functions, the total power constraint need not be considered in the optimization. This is seen if the cost function is written as a function of the signal covariance matrix and the desired beampattern and is of the form

$$\mathcal{C}(\mathbf{R}, I_d) = \int \left| I_d(\theta) - \mathbf{v}^H(\theta)\mathbf{R}\mathbf{v}(\theta) \right|^p w(\theta). \tag{14}$$

We showed in [10] that (14) with piecewise constant $I_d(\theta)$, $w(\theta) = \cos\theta$ and $p = 2$ can be evaluated in closed form for uniform linear arrays. Although the choice of cost function affects the performance of the beampattern optimization, different cost functions are not compared due to lack of space.

Now, if $\mathbf{R}_o$ is a (local) minimum of $\mathcal{C}(\mathbf{R}, I_d)$, then $\alpha\mathbf{R}_o$ is a (local) minimum of $\mathcal{C}(\mathbf{R}, \alpha I_d)$. Therefore, we can minimize the cost with arbitrary scaling of I_d and then find such an α that $\alpha\mathbf{R}_o$ meets the total power constraint. The positive-semidefiniteness of $\mathbf{R}$ is guaranteed by the optimizing the square root $\mathbf{Q}$. In essence, the beampattern is optimized to the shape of the desired beampattern; there is no need to optimize the scale as was done in [7].

We take advantage of the ability to optimize the beampattern without any constraints in the next section, in which the beampattern synthesis is considered as least squares problem.

IV. Beampattern Optimization as Least Squares Problem

In the previous section, beampattern optimization was considered constraining $\mathbf{R}$ to be only positive-semidefinitene. In this section, we show how the beampattern optimization can be represented in least squares form by relaxing this constraint.

In order to optimize using the least squares criterion, we use a cost function

$$\mathcal{C}(\mathbf{R}, I_d) = \sum_{k=1}^{K} \left| I_d(\theta_k) - \mathbf{v}^H(\theta_k)\mathbf{R}\mathbf{v}(\theta_k) \right|^2 w(\theta_k), \tag{15}$$

which can be seen as an approximation of (14) with $p = 2$. To form the LS problem, we need find vectors $\tilde{\mathbf{r}}$ and $\tilde{\mathbf{v}}(\theta)$ such that $\tilde{\mathbf{v}}^T(\theta)\tilde{\mathbf{r}} = \mathbf{v}^H(\theta)\mathbf{R}\mathbf{v}(\theta)$. As was shown in [7], this can be done by constructing a matrix $\mathbf{B}$ such that $\mathrm{vec}(\mathbf{R}) = \mathbf{B}\tilde{\mathbf{r}}$, where $\tilde{\mathbf{r}}$ is an $N^2 \times 1$ real-valued vector consisting of the real and imaginary parts of the elements of $\mathbf{R}$ and $(\mathbf{B})_{mn} \in \{0, 1, \pm j\}$. Thus,

$$\mathbf{v}^H(\theta)\mathbf{R}\mathbf{v}(\theta) = [\mathbf{v}^T(\theta) \otimes \mathbf{v}^H(\theta)]\mathbf{B}\tilde{\mathbf{r}} = \tilde{\mathbf{v}}^T(\theta)\tilde{\mathbf{r}}. \tag{16}$$

Let

$$\tilde{\mathbf{p}} = \begin{bmatrix} I_d(\theta_1) \\ \vdots \\ I_d(\theta_K) \end{bmatrix} \quad \text{and} \quad \tilde{\mathbf{A}} = \begin{bmatrix} \tilde{\mathbf{v}}^T(\theta_1) \\ \vdots \\ \tilde{\mathbf{v}}^T(\theta_K) \end{bmatrix}$$

so that the beampattern error vector is

$$\boldsymbol{\varepsilon} = \tilde{\mathbf{p}} - \tilde{\mathbf{A}}\tilde{\mathbf{r}} \tag{17}$$

and $\mathcal{C}(\mathbf{R}, I_d) = \boldsymbol{\varepsilon}^H \mathbf{W} \boldsymbol{\varepsilon}$ with the minimum

$$\tilde{\mathbf{r}}_o = (\tilde{\mathbf{A}}^T \mathbf{W} \tilde{\mathbf{A}})^{-1} \tilde{\mathbf{A}}^T \mathbf{W} \tilde{\mathbf{p}}, \tag{18}$$

where $\mathbf{W}$ is a diagonal matrix with $(\mathbf{W})_{kk} = w(\theta_k)$. However, resulting $\mathbf{R}$ constructed from $\tilde{\mathbf{r}}_o$ is not valid as it is not positive-semidefinite.

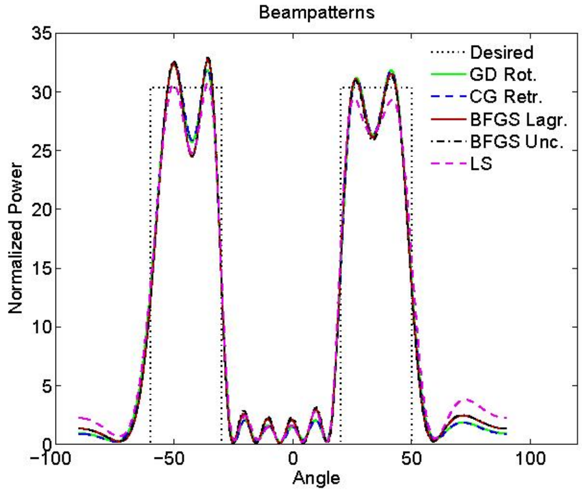
A positive-semidefinite $\mathbf{R}$ can be obtained from the LS solution by using either singular value decomposition or eigenvalue decomposition. If $\mathbf{U}\mathbf{S}\mathbf{V}^H$ is the singular value decomposition of a nondefinite matrix, $\mathbf{V}\mathbf{S}\mathbf{V}^H$ is positive-definite. A rank-deficient covariance matrix can be obtained by omitting some of the singular values and vectors. If the eigenvalue decomposition is used, all negative eigenvalues have to be omitted when constructing the covariance matrix from the eigenvalues and eigenvectors. Therefore, the resulting matrix is generally rank-deficient.

It should be noted that the covariance matrices obtained using the decompositions are not generally solutions to the problem $\min \mathcal{C}(\mathbf{R}, I_d)$ s.t. $\mathbf{R}$ positive-definite.

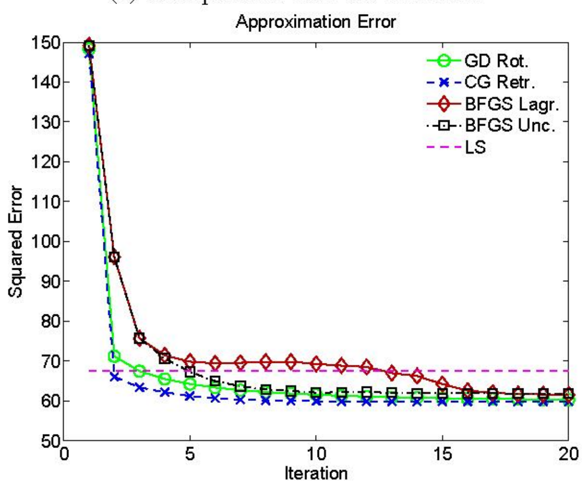
V. Examples

In this section, we show the results of optimizing a beampattern for 12-element ULA with half-wavelength interelement spacing. Five different optimization methods that were discussed in the previous sections are compared. These methods are the gradient descent with rotation (GD Rot.), conjugate gradient with retraction (CG Retr.), Lagrange multipliers using the Broyden-Fletcher-Goldfarb-Shanno (BFGS) method, unconstrained BFGS, and the rank-deficient least squares solution (LS). The cost function used was that of (15) with $w(\theta) \equiv 1$.

The desired beampattern had two separate beams from $-60°$ to $-30°$ and from $20°$ to $50°$. The optimization was done

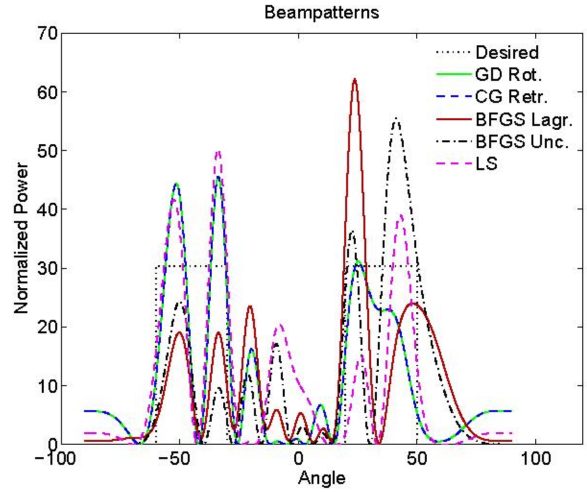


(a) beampatterns after 20 iterations

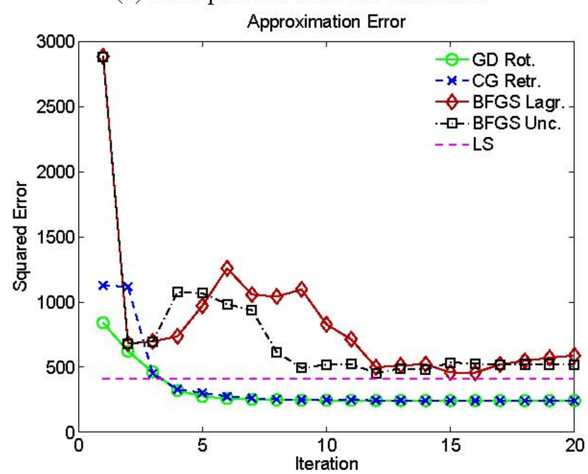


(b) squared approximation error

Fig. 1. Beampatterns with elemental power constraint and full-rank $\mathbf{R}$. The resulting beampatterns are quite similar, although the LS method has slightly higher sidelobes. The conjugate gradient (CG) has best convergence in this case, closely followed by gradient descent (GD).



(a) beampatterns after 20 iterations



(b) squared approximation error

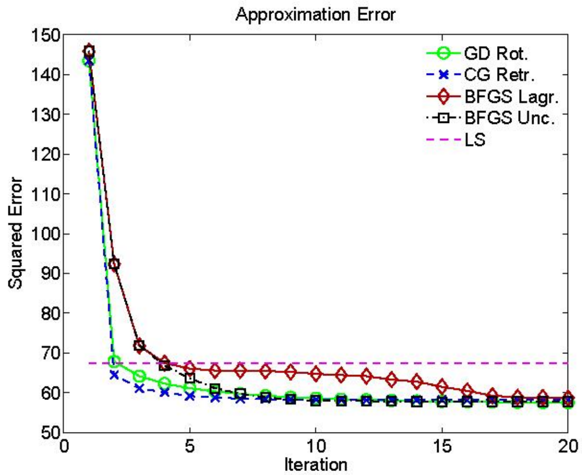
Fig. 2. Beampatterns with elemental power constraint and rank-one $\mathbf{R}$. The desired beampattern is not well suited for a phased array, but the gradient descent (GD) and the conjugate gradient (CG) produce the best beampatterns results. The unconstrained methods suffer from scaling of the weights to satisfy the elemental power constraint. This is also reflected in the convergence of the approximation error.

first with elemental power constraint and full-rank $\mathbf{R}$. The results are shown in Fig. 1, in which Fig.1(a) shows the achieved beampatterns after 20 iterations and 1(b) the squared beampattern approximation error as a function of the iteration. The resulting beampatterns are quite similar, although the LS method has somewhat higher sidelobes than the other methods. The conjugate gradient has the best convergence in this case. Also the gradient descent has similar performance. The convergence of the unconstrained optimization with BFGS in slightly slower than CG and GD. The method of Lagrange multipliers has the slowest convergence among the considered methods.

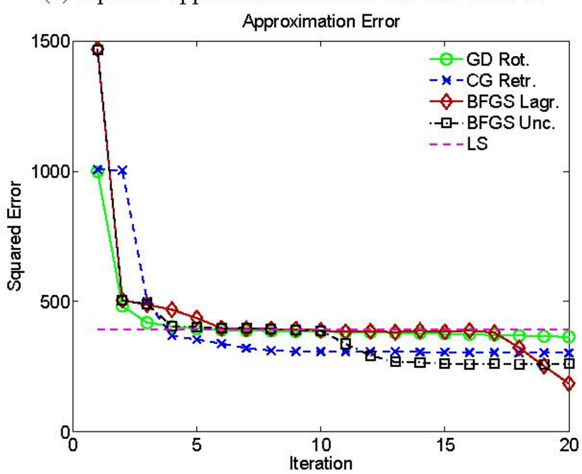
Fig. 2 shows the result of optimizing the beampatterns also with elemental power constraint but rank-one $\mathbf{R}$. The desired beampattern was too complex for rank-one phased array with uniform power, but CG and GD achieved quite reasonable beampatterns. The performance of the unconstrained method is particularly bad in this case, as scaling each element of the $N \times 1$ matrix $\mathbf{Q}$ to have unit modulus changes $\mathbf{R}$ with adverse effect on the optimization.

The squared approximation error for the cases with total power constraint are shown in Figure 3. Fig.3(a) is the full-rank and Fig.3(b) the rank-one case. CG performs slightly better than GD again, but the unconstrained optimization methods also work well in this case.

All in all, the conjugate gradient performed the best of the tested methods. The method of Lagrange multipliers had the slowest convergence and provided good results only when the total power constraint was used. Due to the higher computational cost, it is not practical. The unconstrained optimization provided good results with total power constraint



(a) squared approximation error for full-rank $\mathbf{R}$



(b) squared approximation error for rank-one $\mathbf{R}$

Fig. 3. Beampattern approximation errors with total power constraint. The conjugate gradient and the gradient descent perform well also in these cases, but the performance of the unconstrained methods is good as well as the rows of $\mathbf{Q}$ do not need to be scaled.

but outperformed CG only in the rank-one case. CG had the best overall performance among the considered methods for beampattern optimization that exploits the correlation of the transmitted waveforms.

VI. Conclusions

Beamforming is used in radars to maximize the gain in the direction of the target. A transmit beamforming method exploiting signal correlation was proposed in [1]. This method can be used for shaping the beampattern flexibly.

We have demonstrated in this paper that the beampattern optimization method in [1] can be modified in a very simple fashion to obtain signal covariance matrices with reduced rank. Reducing the rank can be done real time and it also reduces the computational complexity. We also showed how the optimization method can be extended to use the total power constraint in addition to the elemental power constraint.

We also proposed converting the constrained optimization problem into an unconstrained problem. This can be done using the method of Lagrange multipliers, but it was shown that in case of a total power constraint, the optimization can be done without the constraint and the result can be scaled to satisfy the constraint on the total power. Taking advantage of this, we proposed that approximate beampatterns can be obtained by formulating the beampattern optimization problem as a least squares problem that can be solved with low computational complexity.

The use of various optimization methods in the beampattern optimization was studied using numerical examples in Section V. It was seen that the proposed conjugate gradient method with retraction worked well. The results demonstrated that proposed changes in the optimization algorithm and also the unconstrained and the least squares approaches result in feasible beampatterns.

References

[1] D.R. Fuhrmann and G.S. Antonio, "Transmit beamforming for mimo radar systems using partial signal correlation," in *Conference Record of the Thirty-Eighth Asilomar Conference on Signals, Systems and Computers*, 2004, vol. 1, pp. 295–299.

[2] D.J. Rabideau and P. Parker, "Ubiquitous mimo multifunction digital array radar," in *Conference Record of the Thirty-Eighth Asilomar Conference on Signals, Systems and Computers*, 2003, vol. 1, pp. 1057–1064.

[3] D.W. Bliss and K.W. Forsythe, "Multiple-input multiple-output (mimo) radar and imaging: degrees of freedom and resolution," in *Conference Record of the Thirty-Seventh Asilomar Conference on Signals, Systems and Computers*, 2003, vol. 1, pp. 54–59.

[4] F.C. Robey, S. Coutts, D. Weikle, J.C. McHarg, and K. Cuomo, "Mimo radar theory and experimental results," in *Conference Record of the Thirty-Eighth Asilomar Conference on Signals, Systems and Computers*, 2004, vol. 1, pp. 300–304.

[5] E. Fishler, A. Haimovich, R. Blum, D. Chizhik, L. Cimini, and R. Valenzuela, "Mimo radar: an idea whose time has come," in *Proceedings of the IEEE Radar Conference*, 2004, pp. 71–78.

[6] E. Fishler, A. Haimovich, R.S. Blum, L.J. Cimini, D. Chizhik, and R.A. Valenzuela, "Spatial diversity in radars - models and detection performance," *IEEE Transactions on Signal Processing*, vol. 54, no. 3, pp. 823–838, 2006.

[7] Petre Stoica, Jian Li, and Yao Xie, "On probing signal design for mimo radar," *IEEE Transactions on Signal Processing*, vol. 55, no. 8, pp. 4151–4161, Aug 2007.

[8] Pierre-Antoine Absil and Kyle Gallivan, "Joint diagonalization on the oblique manifold for independent component analysis," in *IEEE International Conference on Acoustics, Speech, and Signal Processing*, 2006, vol. 5, pp. 945–948.

[9] S. Smith, "Optimum phase-only adaptive nulling," *IEEE Transactions on Signal Processing*, vol. 47, no. 7, pp. 1835–1843, Jul 1999.

[10] Tuomas Aittomäki and Visa Koivunen, "Low-complexity method for transmit beamforming in mimo radars," in *Proceedings of IEEE International Conference on Acoustics, Speech and Signal Processing*, 2007, vol. 2, pp. 305–308.