Insurance: Mathematics and Economics 80 (2018) 15–28

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Optimal insurance design under background risk with dependence

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ARTICLE INFO

Article history:
Received June 2017
Received in revised form February 2018
Accepted 26 February 2018
Available online 4 March 2018

Keywords:
Optimal insurance
Background risk
Stop-loss order
Expected indemnity
Dependence

ABSTRACT

In this paper, we revisit the problem of optimal insurance under a general criterion that preserves stop-loss order when the insured faces two mutually dependent risks: background risk and insurable risk. According to the local monotonicity of conditional survival function, we derive the optimal contract forms in different types of interval. Because the conditional survival function reflects the dependence between background risk and insurable risk, the dependence structure between the two risks plays a critical role in the insured's optimal insurance design. Furthermore, we obtain the optimal insurance forms explicitly under some special dependence structures. It is shown that deductible insurance is optimal and the Mossin's Theorem is still valid when background risk is stochastically increasing in insurable risk, which generalizes the corresponding results in Lu et al. (2012). Moreover, we show that an individual will purchase no insurance when the sum of the two risks is stochastically decreasing in insurable risk.

1. Introduction

The problem of optimal insurance design has drawn significant interest in both research and practice since the seminal papers by Borch (1960) and Arrow (1963) and lots of models have been formulated and studied extensively. According to the sources of risk assumed in the model, the studies of optimal insurance can be classified into the following two categories: the models with one source of risk and the models with background risk.

The former are based on the assumption that there is only one source of risk which can be entirely insured; see, e.g., the survey paper Bernard (2013), Loubergé (2013), Gollier (2013) and recent literatures such as Carlier and Dana (2003), Karni (2008), Guerra and Centeno (2008), Guerra and Centeno (2010), Chi and Lin (2014), Cui et al. (2013), Gollier (2014), Cheung et al. (2015), Bernard et al. (2015), Golubin (2016), Li and Xu (2017), and the references therein. However, the assumption of one source of risk is a simplification because almost everyone faces risks that can be quantified but cannot be insured, for example, war, floods, market valuation of stocks, inflation and other general economic conditions. These uninsurable risks are generally called background risks.

Most studies have arrived at the same conclusion that the addition of an independent background risk does not alter the form of the optimal insurance derived under the assumption of one source of risk but does affect optimal level (see Briys and Viala (1995), Gollier (1996), Mahul (2000) and Lu et al. (2012)). However, as noted in Briys and Viala (1995), the classical results with one source of risk such as Raviv (1979), do not necessarily hold if there exists positive dependence between the insurable risk and the background risk. Hence, the presence of background risk will affect the optimal form and level of the insurance contract. Exploring optimal insurance design under stochastic background risk led to the second type of optimal insurance problem.

In the past thirty years, numerous papers have been dedicated to studying the optimal insurance policy in the presence of background risk. The studies on this issue can be divided into two categories according to whether the contractual forms are known. Some of the research attempt to determine the optimal level based on the assumption that contractual forms are restricted to specific types of insurance, such as deductible insurance or proportional insurance. For example, Doherty and Schlesinger (1983) have explored the choice of an optimal deductible in an insurance contract when initial wealth is random and show that Mossin's Theorem[1] holds if the correlation between initial wealth and insurable losses is negative or zero, but the principle may not be valid if the correlation is positive. In Jeleva (2000), optimal proportional insurance was analyzed in a nonprobabilized uncertainty framework,

[1] A well-known result in the literature is the Bernoulli principle, which postulates that full coverage is optimal if insurance premiums are actuarially fair. Mossin (1968) extended this result to show that less than full coverage is optimal when the insurance premium includes a positive, proportional loading. In this study, the aforementioned results are generally referred to as Mossin's Theorem.

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https://doi.org/10.1016/j.insmatheco.2018.02.006
0167-6687/

where the preferences are represented using the Choquet expected utility model. Luciano and Kast (2001) studied the effects of an uninsurable background risk on the demand for contractual forms of deductible insurance and proportional insurance under the Mean-VaR framework. To overcome the deficiencies of the classical correlation coefficient, Hong et al. (2011) re-examined Mossin's Theorem under random initial wealth by extending the correlation coefficient to more general dependence measures called positively (negatively) expectation dependent; see also Mayers and Smith (1983), Schlesinger (1997), Meyer and Meyer (1998), Guiso and Jappelli (1998) and references therein.

On the other hand, other studies aim to identify optimal contractual forms of insurance under some given constraints. Briys and Viala (1995) may be the first work on this issue. They extended Raviv's (1979) framework on the design of optimal policy to include the presence of an uninsurable background risk and revealed that the policy with a disappearing deductible is optimal under a specific positive dependence between the insurable risk and the background risk. Gollier (1996) studied the optimal additive insurance model[2] and showed that the optimal insurance policy may be a disappearing deductible if the uninsurable asset increases with the size of the insurable asset under the assumption that the policyholder is prudent. Mahul (2000) re-examined Raviv's (1979) result in the case that the insured faces both background risk and insurable risk. The author obtained a similar result to Gollier (1996) when an increase in the insurable loss makes riskier the policyholder's random initial wealth, according to any degree of stochastic dominance, and if the derivatives of his utility function alternate in sign. Vercammen (2001) revisited the problem of optimal insurance under the assumption that the background risk and the insurable risk are nonseparable with positive dependence. Contrary to the result of the disappearing deductible in Gollier (1996), Vercammen (2001) showed that the optimal contract is coinsurance above a deductible minimum when the insured is prudent. Dana and Scarsini (2007) examined qualitative properties of efficient insurance contracts in the presence of background risk under different assumptions on the stochastic dependence between the insurable and uninsurable risk. Lu et al. (2012) studied the problem of optimal insurance when the insurable risk and uninsurable background risk are positively dependent in the framework of expected utility. They showed that the deductible insurance is optimal and Mossin's Theorem still holds. Furthermore, the shifts of optimal deductible and expected utility by modification of the dependence structure and the marginal are analyzed. Huang et al. (2013) discusses the optimal insurance contract endogenously under the assumption that background risk depends on insurable loss. Based on some specific assumptions, the optimal insurance policies were derived under the expected utility and mean–variance frameworks and the results under different frameworks are compared. In a recent study by Chi (2015), the author investigated the optimal form of insurance under the mean–variance framework by imposing an incentive-compatible constraint on the coverage function. With the help of a constructive approach, Chi (2015) derived the optimal insurance form explicitly, which depends heavily on the conditional expectation function of the background risk with respect to the insurable risk.

For an insurance contract, the optimal form and quantity, from the insured's point of view, depend on his optimization criterion. Many optimization criteria such as expected-utility criterion, variance criterion, adjustment coefficient criterion, the criteria based on the distortion risk measure (including VaR, TVaR) etc., have been proposed for the derivation of the optimal (re)insurance. Nevertheless, as noted by Denuit and Vermandele (1998), most criteria often reduce to the comparison of certain characteristics of the retained risks and cannot been used to take into account all the characteristics of the underlying probability distribution.

During the past forty years, the problem of comparing risks lies at the heart of insurance business and the theory of stochastic order provides a powerful analysis tool in studying the optimal insurance. A more general criterion that preserves stop-order (convex order) has been adopted in some studies of optimal insurance. The criterion that preserves stop-loss order is first formally proposed by Van Heerwaarden et al. (1989). Gollier and Schlesinger (1996) proved Arrow's theorem on the optimality of deductibles under the optimization criterion of second-degree stochastic dominance (i.e. stop-loss order), without invoking the expected-utility hypothesis. Denuit and Vermandele (1998) derived new results about the optimal reinsurance coverage , when the optimality criterion consists in minimizing the retained risk with respect to the stop-loss order. In a recent study on the optimal reinsurance by Chi and Tan (2011), the criterion that preserves convex order has been adopted as optimization criterion. One of the advantages of the criterion that preserves stop-order (convex order) is that most of the probability characteristics of the retained risk can be considered simultaneously. In addition, many usual optimization criteria including expected-utility maximizing criterion, variance minimizing criterion, adjustment coefficient maximizing criterion, TVaR criterion, just name a few, are all special cases of it (see Van Heerwaarden et al. (1989), Denuit and Vermandele (1998), Dhaene et al. (2006), etc.), that is, the optimal insurance policy under the criterion that preserves stop-loss order will also be the most desirable with respect to these criteria.

When we study optimal insurance under background risk with dependence, the structure of dependence between the two risks needs to be considered. Some concepts of dependence between two random variables have been introduced. For example, Hong et al. (2011) re-examined Mossin's Theorem under random initial wealth by replacing the correlation coefficient with the structures of positive (negative) quadrant dependence, stochastic increasing (decreasing) dependence and positive (negative) expectation dependence. Among various notions of dependence, stochastic increasing (decreasing) dependence is interesting to model dependent risks and may be the most common one in the studying of optimal insurance (see Dana and Scarsini (2007), Cai and Wei (2012), Lu et al. (2012), etc.). So in this study, we will still describe the dependence between the insurable risk and background risk by using the concepts of stochastic increasing (decreasing) dependence.

In this paper, we extend the studies of Van Heerwaarden et al. (1989) and Gollier and Schlesinger (1996), and explore optimal insurance in the presence of background risk under a general criterion that preserves stop-loss order. According to the local monotonicity of the conditional survival function, we derive the optimal insurance form on different types of intervals by applying a constructive approach. Furthermore, we obtain the optimal insurance forms explicitly under some special dependence structures. It is shown that deductible insurance is optimal and Mossin's Theorem is still valid when background risk is stochastically increasing in insurable risk, which generalizes the corresponding results in Lu et al. (2012). Moreover, we show that an individual will purchase no insurance when the sum of the two risks is stochastically decreasing in insurable risk. In the case that $Y \uparrow_{st} X$, Lu et al. (2012) and Chi (2015) reached the same conclusion under the framework of expected utility and mean–variance respectively. However, under the framework with expected utility and the assumption that Y and X are positively dependent (not positive increasing dependent), Gollier (1996) and Dana and Scarsini (2007) showed that the optimal contract was a disappearing deductible , which would

[2] The Models of optimal insurance with background risk can also be classified into two groups. The first is concerned with the case where the risks are additive, and the second is related to the case where the risks are not additive.

lead to ex post moral hazard. The reason is due to the absence of incentive compatibility (see Chi (2015)). Under the assumption that $Y + X\downarrow_{st}X$, the same conclusion is obtained by Chi (2015) under the assumption of strongly negative expectation dependent in the framework of mean–variance. However, in this case, Dana and Scarsini (2007) showed that neither no insurance nor full insurance is optimal, and that the optimal ceded loss function is decreasing over some interval. The reason is as the same as that in the previous case.

The remainder of the paper is organized as follows. Section 2 formulates the optimal insurance problem and reviews some basic definitions and preliminary results. In Section 3, we derive the optimal insurance form on different types of intervals according to the local monotonicity of the conditional survival function. In Section 4, the optimal insurance forms are obtained explicitly under some special dependence structures. Section 5 concludes the paper.

2. Preliminaries and the model

We begin by reviewing some basic definitions and properties of stochastic orders and dependence structures between two random variables that are useful in the following.

Definition 2.1. Let X and Y be two random variables.

(i) X is said to be smaller than Y in convex order, denoted by $X\preceq_{cx}Y$, if

$$\mathbb{E}[\phi(X)] \leq \mathbb{E}[\phi(Y)]$$

for any convex function ϕ such that the expectations exist.

(ii) X is said to be smaller than Y in stop-loss order, denoted by $X\preceq_{sl}Y$, if

$$\mathbb{E}[(X-t)_+] \leq \mathbb{E}[(Y-t)_+]$$

for all real t such that the expectations exist.

Lemma 2.2 (*Müller and Stoyan, 2002*)**.** *The following statements are equivalent:*

(i) $X\preceq_{cx}Y$;
(ii) $X\preceq_{sl}Y$ *and* $\mathbb{E}[X] = \mathbb{E}[Y]$.

Note that Stop-loss order is often called "increasing convex order". For details on stochastic order, we refer to Müller and Stoyan (2002), Denuit et al. (2005) and Shaked and Shanthikumar (2007).

There is a variety of concepts of positive and negative dependence between two random variables, for example, positive (negative) quadrant dependence, positive (negative) supermodular dependence, associated dependence, stochastic increasing (decreasing) dependence, positive (negative) expectation dependence, and so on. In the following, we only recall the notions of stochastic increasing (decreasing), which will be used in this paper, For other notions of positive dependence, we refer to Lehmann (1966), Joe (1997), Colangelo et al. (2005), Denuit et al. (2005), Nelsen (2006), Shaked and Shanthikumar (2007) and references therein.

Definition 2.3. Let X and Y be two random variables.

(i) Y is said to be (strictly) stochastically increasing in X, denoted by $Y\uparrow_{st}X$ ($Y\uparrow_{sst}X$), if

$$\mathbb{P}(Y > y|X = x)$$

is a (strictly) increasing function of x for all $x \in Supp(X)$, or equivalently,

$$\mathbb{E}[u(Y)|X = x]$$

is (strictly) increasing in $x \in Supp(X)$ for all (strictly) increasing function u such that the expectations exist, where $Supp(X)$ is the support of X.

(ii) Y is said to be (strictly) stochastically decreasing in X, denoted by $Y\downarrow_{st}X$ ($Y\downarrow_{sst}X$), if

$$\mathbb{P}(Y > y|X = x)$$

is a (strictly) decreasing function of x for all $x \in Supp(X)$, or equivalently,

$$\mathbb{E}[u(Y)|X = x]$$

is (strictly) decreasing in $x \in Supp(X)$ for all (strictly) increasing functions u such that the expectations exist.

Note that the concept of stochastically increasing (decreasing) is also called "positive (negative) regression dependent" in Lehmann (1966). For two random variables X and Y, an extreme case of stochastic increasing (decreasing) dependence is the one that X and Y are comonotonic (anti-comonotonic). Note that stochastic increasing (decreasing) dependence is an asymmetric concept, that is, the fact that Y is stochastically increasing in X does not imply that X is stochastically increasing in Y. The notion of stochastic increasing (decreasing) dependence is interesting to model dependent risks and has been widely used in many fields such as the theory of reliability (Barlow and Proschan, 1975), risk management with dependence (Denuit et al., 2005), etc. In the studying of optimal insurance under multiple sources of risk, it has always been assumed that the dependence structures among risks are stochastically increasing (decreasing), see, e.g., Dana and Scarsini (2007), Hong et al. (2011), Cai and Wei (2012) and Lu et al. (2012).

Let random variables X and Y, defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, denote two sources of risk faced by an individual during some fixed time, where X is insurable and non-negative, while Y is the background risk and may be negative. We further assume that X and Y are absolutely continuous. F, S and f are used to denote the distribution function, survival function and density function of a random variable, respectively.

To reduce the risk faced, the insured would seek protection by purchasing an insurance policy. Denote the coverage function by $I(x)$, which represents the payment received from the insurer when loss x occurs under the insurance contract. We assume that $I(x)$ is increasing with $I(0) = 0$ and $0 \leq I(x) \leq x$, which are always used in the studies of optimal insurance. However, as noted in Chi (2015) and Cheung (2010), it is not sufficient to study the problems of optimal insurance, as ex post moral hazard may arise under the contracts that follow this assumption. To avoid moral issues, we further suppose that the coverage function is 1-Lipschitz (called incentive-compatible constraint), i.e.

$$I(x_2) - I(x_1) \leq x_2 - x_1 \tag{2.1}$$

for any $0 \leq x_1 \leq x_2$, which was first proposed by Huberman et al. (1983). The collection of all coverage functions satisfying the assumptions above is denoted by $\mathscr{I}$.

Underwriting part of potential loss for the insured, the insurer will be compensated by the insurance premium. Denote by $\Pi_I(X)$ the insurance premium that corresponds to the coverage $I(x)$. As in Gollier (1996), we assume that the premium is based upon the expected indemnity:

$$\Pi_I(X) = \pi(\mathbb{E}[I(X)])$$

where π is a differentiable function with $\pi(0) = 0$ and $\pi'(x) \geq 1$ for any $x \geq 0$. If $\pi(x) = (1+\theta)x$, then the expected indemnity reduced to the common expectation value premium principle, where $\theta \geq 0$ is the safety loading factor.

Under insurance contract I, the random total risk exposure of insured, denoted by T_I, is

$$T_I = Y + X - I(X) + \Pi_I(X).$$

As mentioned in Denuit and Vermandele (1998), most criteria in the problem of optimal insurance often reduce to the comparison of certain characteristics of the retained risks and cannot be used to take into account all the characteristics of the underlying probability distribution. The criterion that preserves stop-order (convex order), which includes many common criteria as its special cases, can consider most of the probability characteristics of the retained risk simultaneously.

It is well known that a necessary condition of convex order between two random variables is that they have the same expectation. Otherwise, we cannot order them in the sense of convex order. In general, the study of optimal insurance consists of two contents: the derivation of optimal contractual form and the analysis of transferred quantity. The optimal insurance forms are often derived by keeping the insurance premium fixed. Especially, when the premium principle is based on the expectation of the coverage functions, the expectations are always forced to be a constant. Hence, the theory of convex order is only applicable to derive the optimal contractual form. If we aim to discuss the optimal transferred quantity, it will fail because the insurance premium will also be a variable. To overcome this drawback, in this paper, we will adopt the optimization criterion that is based on stop-loss order. Specifically, we are seeking the optimal insurance policy $I \in \mathscr{I}$ that minimize the risk measure of the total risk exposure of the insured:

$$\min_{I\in\mathscr{I}} \rho(T_I), \tag{2.2}$$

where ρ is a risk measure that preserves stop-loss order, i.e.

$$\rho(Z_1) \le \rho(Z_2), \ \text{if}\ Z_1 \preceq_{sl} Z_2.$$

Throughout the paper, increasing and decreasing are used in the non-strict sense; that is, increasing means non-decreasing and decreasing means non-increasing.

3. Optimal insurance design

We begin this section by introducing some notations used in what follows.

For a given $I(x) \in \mathscr{I}$, define

$$\eta_I(x; t) \triangleq \mathbb{P}(Y + X - I(X) > t \mid X = x).$$

$\eta_I(x; t)$ is a special form of survival function, and will play a key role in the following analysis. Being the substitution of distribution function, the survival function has extensive application in applied probability such as in the theory of reliability (see Barlow and Proschan (1975)) and in actuarial science (see Denuit et al. (2005)). As proved in Section 4, for any $I(x) \in \mathscr{I}$, if Y is stochastically increasing in X, then $\eta_I(x; t)$ is increasing with respect to x and if $Y + X$ is stochastically decreasing in X, then $\eta_I(x; t)$ is decreasing with respect to x. Therefore, for a given $I(x)$, the monotonicity of $\eta_I(x; t)$ reflects the dependence structure between risks X and Y and can be seen as a generalization of stochastically increasing (decreasing) dependence in the study of optimal insurance. As mentioned above, the dependence structure of stochastic increasing (decreasing) is the most common one in studying of optimal insurance. Hence the introduction of $\eta_I(x; t)$ is interesting to model dependent risks .

Note that $\eta_I(x; t)$ may be not differentiable. To facilitate the discussion, we will subsequently call $\eta_I(x; t)$ the conditional survival function with respect to $I(x)$.

For any function $\psi(\cdot)$ and interval $[a, b) \subset [0, \infty)$, define

$$\begin{aligned}\mathbb{E}[\psi(X); X \in [a, b)] &\triangleq \mathbb{E}[\psi(X)\mathbb{I}_{\{X\in[a,b)\}}] = \int_a^b \psi(x)dF_X(x) \\ &= \mathbb{P}(X \in [a, b))\mathbb{E}[\psi(X) \mid X \in [a, b)],\end{aligned} \tag{3.1}$$

where $\mathbb{I}$ is the indicator function. It is easy to verify that for any $[a, b), [b, c) \subset [0, \infty)$,

$$\mathbb{E}[\psi(X); X \in [a, b)] + \mathbb{E}[\psi(X); X \in [b, c)] = \mathbb{E}[\psi(X); X \in [a, c)].$$

For any $I(x) \in \mathscr{I}$, we further define

$$\Pi_I(X; X \in [a, b)) \triangleq \pi(\mathbb{E}[I(X); X \in [a, b)]). \tag{3.2}$$

To facilitate the discussion below, we now define some set of functions as follows. For any $I(x) \in \mathscr{I}$ and $a, b \in [0, \infty)$, let

(1) $\mathscr{G}_I^{(1)}$ be the class of non-negative functions $g_1(x; l)$ defined on $[a, b)$ with

$$g_1(x; l) \triangleq I(a) + (x - l)_+ - (x - l - I(b) + I(a))_+, \tag{3.3}$$

where $l \in [a, b - I(b) + I(a)]$.

(2) $\mathscr{G}_I^{(2)}$ be the class of non-negative functions $g_2(x; l)$ defined on $[a, \infty)$ with

$$g_2(x; l) \triangleq I(a) + (x - l)_+,$$

where $l \in [a, \infty)$.

(3) $\mathscr{H}_I^{(1)}$ be the class of non-negative functions $h_1(x; l)$ defined on $[a, b)$ with

$$h_1(x; l) \triangleq I(a) + x - a - (x - l)_+ + (x - l - b + a + I(b) - I(a))_+,$$

where $l \in [a, a + I(b) - I(a)]$.

(4) $\mathscr{H}_I^{(2)}$ be the class of non-negative functions $h_2(x; l)$ defined on $[a, \infty)$ with

$$h_2(x; l) \triangleq I(a) + x - a - (x - l)_+,$$

where $l \in [a, \infty)$.

It is straightforward to check that $\mathscr{G}_I^{(i)}, \mathscr{H}_I^{(i)} \in \mathscr{I}, i = 1, 2$. The curves of these functions are shown in Figs. 3.1 and 3.2.

The following theorem is the key result of this paper, whose proof is presented in the Appendix.

Theorem 3.1. *For a given $I(x) \in \mathscr{I}$ and $[a, b) \subset [0, \infty)$ with $\Pi_I(X; X \in [a, b)) = \Pi_0$, it holds that*

(i) *There exists a function $g_1^*(x) \triangleq g_1(x; l^*) \in \mathscr{G}_I^{(1)}$ such that*

$$\Pi_{g_1^*}(X; X \in [a, b)) = \Pi_0.$$

In addition, if for any $t \in \mathbb{R}$, $\eta_{g_1^*}(x; t)$ is increasing with respect to x in $[a, b)$, then

$$\mathbb{E}[(Tg_1^* - t)_+; X \in [a, b)] \le \mathbb{E}[(T_I - t)_+; X \in [a, b)], \quad \forall\, t \in \mathbb{R}. \tag{3.4}$$

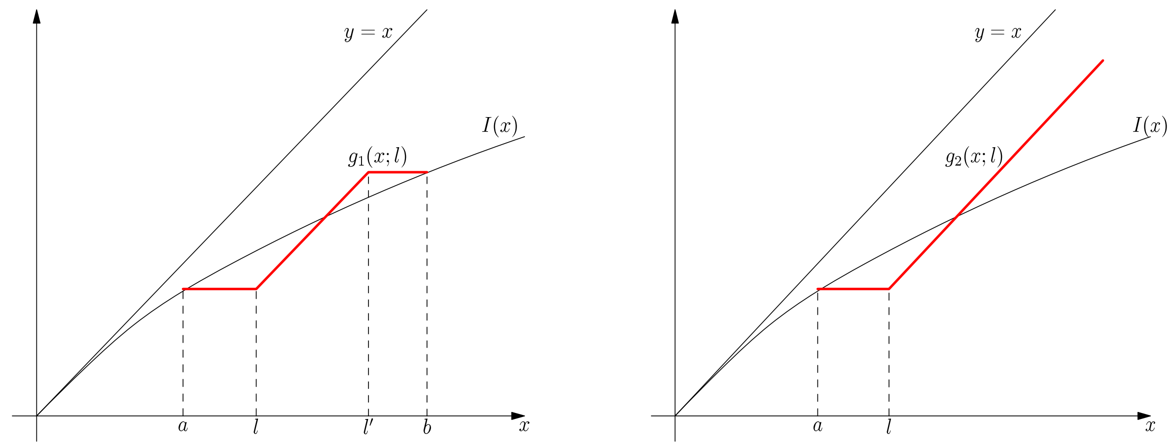
(ii) *There exists a function $h_1^*(x) \triangleq h_1(x; l^*) \in \mathscr{H}_I^{(1)}$ such that*

$$\Pi_{h_1^*}(X; X \in [a, b)) = \Pi_0.$$

In addition, if for any $t \in \mathbb{R}$, $\eta_{h_1^*}(x; t)$ is decreasing with respect to x in $[a, b)$, then

$$\mathbb{E}[(T_{h_1^*} - t)_+; X \in [a, b)] \le \mathbb{E}[(T_I - t)_+; X \in [a, b)], \quad \forall\, t \in \mathbb{R}. \tag{3.5}$$

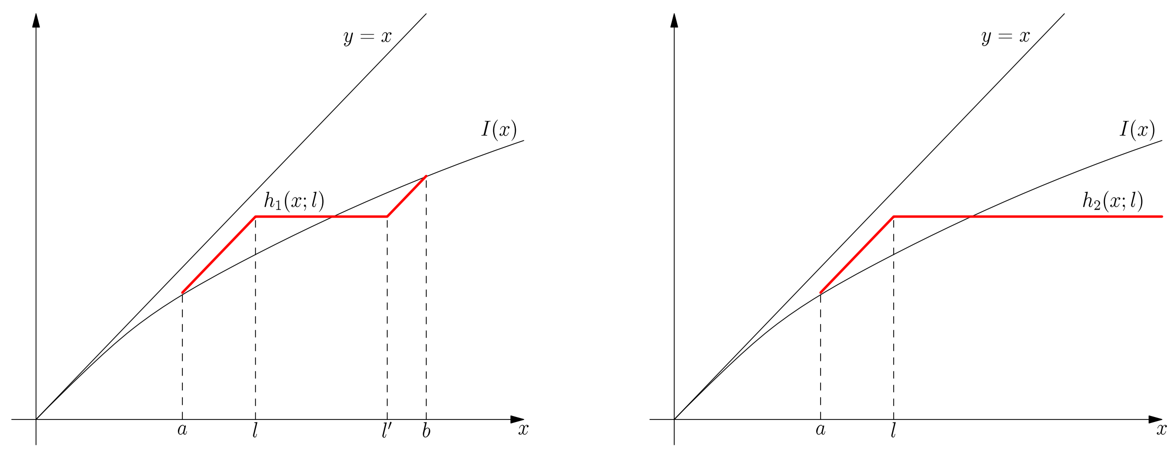
The results show that if the conditional survival function satisfies certain local monotonicity, then any feasible indemnity schedule is dominated by a contract that belongs to the family $\mathscr{G}_I^{(1)}$ (or $\mathscr{H}_I^{(1)}$). Hence, we can derive the optimal insurance policies by



(a) $g_1(x)(l' = l + I(b) - I(a))$.

(b) $g_2(x)$.

Fig. 3.1. The curves of $g_i(x; l)$, $i = 1, 2$.



(c) $h_1(x)(l' = l + b - a - I(b) + I(a))$.

(d) $h_2(x)$.

Fig. 3.2. The curves of $h_i(x; l)$, $i = 1, 2$.

confining the coverage functions within the class $\mathscr{G}_I^{(1)}$ (or $\mathscr{H}_I^{(1)}$), and the problem is reduced to a one-dimensional optimization problem.

Replacing $[a, b)$ in Theorem 3.1 by $[a, \infty)$, we can obtain the following results immediately.

Corollary 3.2. *For a given $I(x) \in \mathscr{I}$ and $a \geq 0$ with $\Pi_I(X; X \in [a, \infty)) = \Pi_0$, it holds that*

(i) *There exists a function $g_2^*(x) \triangleq g_2(x; l^*) \in \mathscr{G}_I^{(2)}$ such that*

$$\Pi_{g_2^*}(X; X \in [a, \infty)) = \Pi_0.$$

In addition, if for any $t \in \mathbb{R}$, $\eta_{g_2^*}(x; t)$ is increasing with respect to x in $[a, \infty)$, then

$$\mathbb{E}[(Tg_2^* - t)_+; X \in [a, \infty)] \leq \mathbb{E}[(T_I - t)_+; X \in [a, \infty)], \quad \forall\, t \in \mathbb{R}.$$

(ii) *There exists a function $h_2^*(x) \triangleq h_2(x; l^*) \in \mathscr{H}_I^{(2)}$ such that*

$$\Pi_{h_2^*}(X; X \in [a, \infty)) = \Pi_0.$$

In addition, if for any $t \in \mathbb{R}$, $\eta_{h_2^*}(x; t)$ is decreasing with respect to x in $[a, \infty)$, then

$$\mathbb{E}[(T_{h_2^*} - t)_+; X \in [a, \infty)] \leq \mathbb{E}[(T_I - t)_+; X \in [a, \infty)], \quad \forall\, t \in \mathbb{R}.$$

From Theorem 3.1 and Corollary 3.2, we can conclude that for a given $I(x) \in \mathscr{I}$, if the set $[0, \infty)$ can be divided into several disjoint subintervals such that in each of the subintervals, the conditional survival function of $g_i(x; t)(h_i(x; t))(i = 1, 2)$ is increasing (decreasing), then there exists a $\tilde{I}(x) \in \mathscr{I}$ such that $T_{\tilde{I}}$ is superior than T_I in the stop-loss order with

$$\mathbb{E}[I(X)] = \mathbb{E}[\tilde{I}(X)].$$

We will illustrate this result by the following example.

Example 3.3. Assume that the conditional distribution of Y given $X = x(x \geq 0)$ is $N(|1 - x|, 1)$. Let $\Phi(\cdot)$ and $\phi(\cdot)$ be the distribution and density function of the standard normal distribution respectively. To proceed, we now consider the following two cases.

(i) $x \in [0, 1)$
From Theorem 3.1, for any $I(x) \in \mathscr{I}$, there exists

$$h_1(x; l_1) = x - (x - l_1)_+ + (x - l'_1)_+ \in \mathscr{H}_I^{(1)}$$

such that $\Pi_{h_1}(X; X \in [0, 1)) = \Pi_I(X; X \in [0, 1))$, where $l_1 \in [0, I(1))$ and $l'_1 = l_1 + 1 - I(1)$.

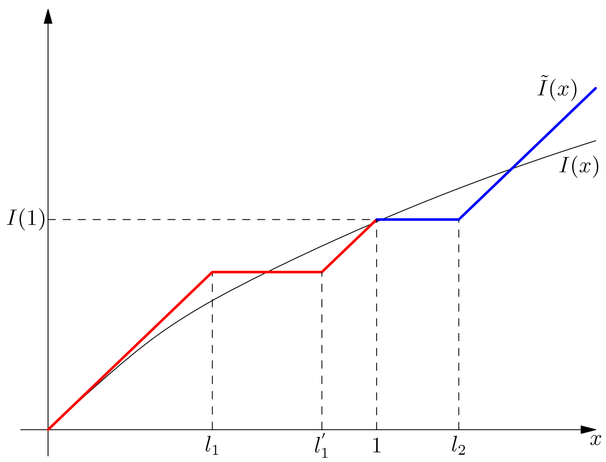


Fig. 3.3. The curves of $I(x)$ and $\tilde{I}(X)$.

Furthermore, for any real t, we have

$$\begin{aligned}\eta_{h_1}(x;t) &= \mathbb{P}(Y > t - x + h_1(x;l_1)|X = x)\\ &= \mathbb{P}(Y > t - (x-l_1)_+ + (x-l_1')_+|X = x)\\ &= 1 - \mathbb{P}(Y \le t - (x-l_1)_+ + (x-l_1')_+|X = x)\\ &= 1 - \Phi\left(t - (x-l_1)_+ + (x-l_1')_+ - 1 + x\right).\end{aligned} \tag{3.6}$$

It is easy to check that for any t, $\eta_{h_1}(x;t)$ is continuous and decreasing with respect to x in $[0, l_1)$ and $[l_1', 1)$ but constant in $[l_1, l_1')$. Hence, $\eta_{h_1}(x;t)$ is decreasing with respect to x in $[0, 1)$.

(ii) $x \in [1, \infty)$

It follows from Corollary 3.2 that for any $I(x) \in \mathscr{I}$, there exists

$$g_2(x;l_2) = I(1) + (x-l_2)_+ \in \mathscr{G}_I^{(2)}$$

with $l_2 \in [1,\infty)$, such that $\Pi_{g_2}(X; X \in [1,\infty)) = \Pi_I(X; X \in [1,\infty))$.

For any real t, we can obtain that

$$\begin{aligned}\eta_{g_2}(x;t) &= \mathbb{P}(Y > t - x + g_2(x;l_2)|X = x)\\ &= \mathbb{P}(Y > t - x + I(1) + (x-l_2)_+|X = x)\\ &= 1 - \mathbb{P}(Y \le t - x + I(1) + (x-l_2)_+|X = x)\\ &= 1 - \Phi\left(t - 2x + 1 + I(1) + (x-l_2)_+\right).\end{aligned} \tag{3.7}$$

One can verify that for any t, $\eta_{g_2}(x;t)$ is increasing with respect to x in $[1,\infty)$.

Then from Theorem 3.1 and Corollary 3.2, for any $I(x) \in \mathscr{I}$, we can construct a $\tilde{I}(x) \in \mathscr{I}$ as follows

$$\tilde{I}(x) = \begin{cases} x - (x-l_1)_+ + (x-l_1')_+, & 0 \le x \le 1,\\ I(1) + (x-l_2)_+, & x > 1,\end{cases} \tag{3.8}$$

such that

$$\Pi_{\tilde{I}}(X; X \in [0,\infty)) = \Pi_I(X; X \in [0,\infty))$$

and

$$\mathbb{E}[(Y + X - \tilde{I}(X) - t)_+] - \mathbb{E}[(Y + X - I(X) - t)_+] \le 0.$$

The curves of $I(x)$ and $\tilde{I}(X)$ are shown in Fig. 3.3.

4. Optimal insurance under special dependence structure

In this section, we study optimal insurance design under some special dependence structure between the two risks. Specifically, we will consider the following two special dependence structures: (a) $Y \uparrow_{st} X$; (b) $Y + X \downarrow_{st} X$.

4.1. The case that $Y \uparrow_{st} X$

Roughly speaking, two risks are positively dependent if the value of them change in the same direction. In the reality insurance practice, an insured always faces both insurable and background risks with positive dependence. For example, one half of a couple purchase health insurance for herself, but the other does not. When a viral influenza comes, the couple often communicate colds to each other. Hence, there exists positive dependence between the health risks of the couple. In another example, an individual insure her residence against fire, but the antiques in her house cannot be insured. In the event of a fire, the larger values of loss of the house are probabilistically associated with the larger one of the antiques.

By virtue of Theorem 3.1, we can easily derive the optimal treaty under the assumption that $Y \uparrow_{st} X$ in the following result.

Proposition 4.1. *If Y is stochastically increasing in X ($Y \uparrow_{st} X$), then the deductible insurance is optimal.*

Proof. For any $I(x) \in \mathscr{I}$, $t \in \mathbb{R}$ and $0 \le x_1 \le x_2$, it follows from (2.1) that

$$x_1 - I(x_1) \le x_2 - I(x_2) \tag{4.1}$$

and

$$\begin{aligned}\eta_I(x_1;t) &= \mathbb{P}(Y + X - I(X) > t|\, X = x_1)\\ &= \mathbb{P}(Y > t - x_1 + I(x_1)|\, X = x_1)\\ &\le \mathbb{P}(Y > t - x_2 + I(x_2)|\, X = x_1)\\ &\le \mathbb{P}(Y > t - x_2 + I(x_2)|\, X = x_2)\\ &= \eta_I(x_2;t),\end{aligned}$$

where the first inequality follows from (4.1) and the second one follows from the assumption that $Y \uparrow_{st} X$. Hence $\eta_I(x;t)$ is increasing. By using Corollary 3.2, there exists $I^*(x) \triangleq (x-l)_+$ such that

$$\mathbb{E}[(T_{I^*} - t)_+] \le \mathbb{E}[(T_I - t)_+],\ \forall\, t \in \mathbb{R},$$

where l can be determined by $\Pi_{I^*}(X; X \in [0,\infty)) = \Pi_I(X; X \in [0,\infty))$, which implies that $T_{I^*} \preceq_{sl} T_I$, as desired. □

Note that the optimal insurance in the case that $Y \uparrow_{st} X$ have been studied by Lu et al. (2012) in the framework of expected utility. The above result generalizes Theorem 3.1 in Lu et al. (2012) since expected utility when the agent is prudent a risk measure that preserves stop-loss order. Meanwhile, Chi (2015) reached the same conclusion under the mean–variance framework by imposing the incentive-compatible constraints on $I(x)$. However, under the framework with expected utility and the assumption that Y and X are positively dependent, Gollier (1996) and Dana and Scarsini (2007) showed that the optimal contract was a disappearing deductible , which would lead to ex post moral hazard; that was because of the absence of incentive-compatibility constraints.

Under insurance policy $I^*(x) \triangleq (x-l)_+$, the random total risk exposure of the insured is given by

$$T_{I^*}(l) = Y + X - (X-l)_+ + \pi\left(\int_l^\infty S_X(x)dx\right).$$

It is easy to verify that $\mathbb{E}\left[(T_{I^*} - t)_+\right]$ can be rewritten as

$$\begin{aligned}\mathbb{E}\left[(T_{I^*}(l) - t)_+\right] &= \int_0^l \left[\int_{\tau(x;l,t)}^\infty S_{Y|X=x}(y)dy\right] f_X(x)dx\\ &\quad + \int_l^\infty \left[\int_{\tau(l;l,t)}^\infty S_{Y|X=x}(y)dy\right] f_X(x)dx\end{aligned} \tag{4.2}$$

with $\tau(u;l,t) = t - u - \pi\left(\int_l^\infty S_X(x)dx\right)$.

For given $t \in \mathbb{R}$, by taking the derivative of (4.2) with respect to l, we obtain that

$$\begin{aligned}&\frac{d}{dl}\mathbb{E}\left[(T_{I^*}(l)-t)_+\right]\\&=-S_X(l)\pi'\left(\int_l^\infty S_X(x)dx\right)\int_0^l S_{Y|X=x}(\tau(x;l,t))f_X(x)dx \qquad (4.3)\\&+\left(1-S_X(l)\pi'\left(\int_l^\infty S_X(x)dx\right)\right)\int_l^\infty S_{Y|X=x}(\tau(l;l,t))f_X(x)dx.\end{aligned}$$

The determination of optimal deductible, denoted by l^*, depends on the risk measure ρ. There are no explicit formulas for the calculation of it. But we still get the following result.

Proposition 4.2. *If $Y\uparrow_{st}X$ and $\pi'(x) = 1$ for any $x \geq 0$, then the optimal deductible is $l^* = 0$.*

Proof. If $\pi'(x) = 1$ for $x \geq 0$, then it holds that $\pi(x) = x+b$, where b is any real number. It follows from (4.3) that

$$\begin{aligned}\frac{d}{dl}\mathbb{E}\left[(T_{I^*}(l)-t)_+\right] &= -S_X(l)\int_0^l S_{Y|X=x}(\tau_0(x;l,t))f_X(x)dx \\ &\quad + (1-S_X(l))\int_l^\infty S_{Y|X=x}(\tau_0(l;l,t))f_X(x)dx,\end{aligned} \qquad (4.4)$$

where $\tau_0(u;l,t) = t-b-u-\int_l^\infty S_X(x)dx$. It is clear that $\frac{d}{dl}\mathbb{E}\left[(T_{I^*}(l)-t)_+\right]|_{l=0} = 0$.

Since $Y\uparrow_{st}X$, for $0 \leq x_1 < x_2$, we have

$$\begin{aligned}S_{Y|X=x_1}(\tau_0(x_1;l,t)) &= \mathbb{P}(Y > t-b-x_1-\int_l^\infty S_X(x)dx|X=x_1)\\&\leq \mathbb{P}(Y > t-b-x_1-\int_l^\infty S_X(x)dx|X=x_2)\\&\leq \mathbb{P}(Y > t-b-x_2-\int_l^\infty S_X(x)dx|X=x_2)\\&= S_{Y|X=x_2}(\tau_0(x_2;l,t)),\ \forall\, t\in\mathbb{R},\end{aligned}$$

which implies that $S_{Y|X=x}(\tau_0(x;l,t))$ is increasing in x.

When $l \neq 0$, by applying the second mean value theorem for integrals, we have

$$\begin{aligned}&\frac{d}{dl}\mathbb{E}\left[(T_{I^*}(l)-t)_+\right]\\&=-S_X(l)\int_0^l S_{Y|X=x}(\tau_0(x;l,t))f_X(x)dx+(1-S_X(l))\\&\quad\times\int_l^\infty S_{Y|X=x}(\tau_0(l;l,t))f_X(x)dx\\&=-S_X(l)S_{Y|X=l}(\tau_0(l;l,t))\int_\zeta^l f_X(x)dx+(1-S_X(l))\\&\quad\times v\int_l^\infty S_{Y|X=x}(\tau_0(l;l,t))f_X(x)dx\\&=-S_X(l)S_{Y|X=l}(\tau_0(l;l,t))\left(1-\int_0^\zeta f_X(x)dx-\int_l^\infty f_X(x)dx\right)\\&\quad+(1-S_X(l))\int_l^\infty S_{Y|X=x}(\tau_0(l;l,t))f_X(x)dx\\&=(1-S_X(l))\int_l^\infty S_{Y|X=x}(\tau_0(l;l,t))-S_{Y|X=l}(\tau_0(l;l,t))f_X(x)dx\\&\quad+S_X(l)S_{Y|X=l}(\tau_0(l;l,t))F_X(\zeta)>0,\end{aligned}$$

where $\zeta \in (0,l)$ and the last equality follows from the assumption that $Y\uparrow_{st}X$. Therefore, $\mathbb{E}\left[(T_{I^*}(l)-t)_+\right]$ must attain its minimum at $l=0$ for any $t\in\mathbb{R}$, as desired. □

Proposition 4.2 shows that if the premium principle is a linear function of its expected value with slope 1, then the optimal insurance policy is full insurance. It was shown in Mossin (1968) that a risk averse individual would choose to fully insure at actuarially fair premium (called the Mossin's Theorem). The Mossin's Theorem can also be obtained in our model since actuarially fair premium satisfies the condition in Proposition 4.2. Hence Proposition 4.2 is a generalization of the Mossin's Theorem.

Proposition 4.3. *If $Y\uparrow_{st}X$ and $\pi'(x) > 1$ for any $x \geq 0$, then the optimal deductible $l^* > 0$.*

Proof. It follows from (4.3) that when $\pi'(x) > 1$ for any $x \geq 0$,

$$\begin{aligned}&\frac{d}{dl}\mathbb{E}\left[(T_{I^*}(l)-t)_+\right]|_{l=0}\\&\quad=\left(1-\pi'(\mathbb{E}[X])\right)\int_0^\infty S_{Y|X=x}(\tau(0;0,t))f_X(x)dx<0,\ \forall\, t\in\mathbb{R},\end{aligned}$$

which implies that there exists some $c > 0$ such that

$$\mathbb{E}\left[(T_{I^*}(l)-t)_+\right]|_{l=c} < \mathbb{E}\left[(T_{I^*}(l)-t)_+\right]|_{l=0},\ \forall\, t\in\mathbb{R}.$$

Hence, we have

$$\rho(T_{I^*}(c)) < \rho(T_{I^*}(0)),$$

which implies that the minimum of $\rho(T_{I^*}(l))$ must be attained at $l^* > 0$. □

The assumption $\pi'(x) > 1$ means that there exists extra costs in addition to the fair premium. So Proposition 4.3 reveals that the partial insurance is optimal when the premium exceeds the corresponding fair premium. Hence we can conclude that extra costs explain deductibles.

Remark 4.4. If the insurance premium in our model is calculated under the common expectation premium, i.e. $\Pi_I(X) = (1+\theta)\mathbb{E}[I(X)]$, then from Proposition 4.2, we can obtain that if $\theta = 0$, then $l^* = 0$. Meanwhile, it follows from Proposition 4.3 that if $\theta > 0$, then $l^* > 0$. Recall that Lu et al. (2012) reach the same conclusion under the framework of expected-utility. Since as a risk measure, the expected-utility preserves the stop-loss order, Proposition 3.5 and Corollary 3.3 of Lu et al. (2012) are the consequences of Proposition 4.2 and Proposition 4.3 respectively.

Remark 4.5. By Definition 2.3, if Y and X are independent, then Y is stochastically increasing in X. Under this special case, the optimal insurance policy is still the deductible insurance and the results obtained above still hold true. Furthermore, the results above give a further verification that the addition of an independent background risk will not always alter the form of the optimal insurance, but may affect the quantity of the optimal insurance. Note that the same result has been reached by Gollier (1996), Mahul (2000) and Huang et al. (2013) under the framework of expected utility.

Example 4.6. Let

$$\rho(X) = \mathbb{E}\left[e^{\frac{X}{2}}\right].$$

It is easy to check that ρ is a risk measure which preserves the stop-loss order since the function $e^{\frac{x}{2}}$ is an increasing convex function. In the economic sense, ρ can be regarded as a risk measure induced by the expected utility with a negative exponential utility function. In fact, if we suppose that a policyholder has initial wealth ω and a negative exponential utility function $u = -2e^{-\frac{1}{2}x}$, which means that the policyholder is a risk averter. Then from Section 2, we know that under insurance contract I, the expected utility of final wealth of the insured is

$$\mathbb{E}[u(W)] = -2\mathbb{E}[e^{-\frac{1}{2}(\omega - Y - X + I(X) - \Pi_I(X))}]$$
$$= -2e^{-\frac{1}{2}\omega}\mathbb{E}[e^{\frac{1}{2}(Y+X-I(X)+\Pi_I(X))}]$$
$$= -2e^{-\frac{1}{2}\omega}\mathbb{E}[e^{\frac{1}{2}T_I}] = -2e^{-\frac{1}{2}\omega}\rho(T_I).$$

Under a insurance contract, the target of the policyholder is to maximize his expected utility, which is equivalent to minimizing his risk of the total exposure under risk measure ρ.

Suppose that (X, Y) has a copula C_α who is a member of the Farlie–Gumbel–Morgenstern family copulas, i.e.

$$C_\alpha(u, v) = uv + \alpha uv(1-u)(1-v), \tag{4.5}$$

where $\alpha \in [0, 1]$. From example 5.2 and 5.7 of Nelsen (2006), α can be seen as a dependent parameter (see also chapter 2 of Joe (1997)) because the amount of dependence is increasing as α increases.

We further assume that the insurance premium is given by $\Pi_I(X) = (1+\theta)\mathbb{E}[I(X)]$ with $0 \leq \theta \leq 1$ and both the marginal distributions are exponential distributions and their common distribution function is given by

$$F(x) = 1 - e^{-x}, x \geq 0. \tag{4.6}$$

Then the joint distribution function and density function of (X, Y) are

$$F(x, y) = (1-e^{-x})(1-e^{-y})(1+\alpha e^{-x}e^{-y}), x \geq 0, y \geq 0 \tag{4.7}$$

and

$$f(x, y) = e^{-x}e^{-y} + \alpha(2e^{-2x} - e^{-x})(2e^{-2y} - e^{-y}),\quad x \geq 0, y \geq 0 \tag{4.8}$$

respectively.

Furthermore, the conditional density function and distribution function of Y given $X = x > 0$ can be obtained as follows:

$$f_{Y|X}(y|x) = e^{-y} + \alpha(2e^{-x} - 1)(2e^{-2y} - e^{-y}), y \geq 0 \tag{4.9}$$

and

$$F_{Y|X}(y|x) = 1 - e^{-y} + \alpha(2e^{-x} - 1)(e^{-y} - e^{-2y}), y \geq 0. \tag{4.10}$$

Then we have

$$\mathbb{P}(Y > y|X = x) = e^{-y} - \alpha(2e^{-x} - 1)(e^{-y} - e^{-2y}). \tag{4.11}$$

By taking the derivative of (4.11) with respect to x, we get

$$\frac{\partial}{\partial x}\mathbb{P}(Y > y|X = x) = 2\alpha e^{-x}(e^{-y} - e^{-2y}) \geq 0,$$

which implies that $Y \uparrow_{st} X$ since $\alpha \geq 0$.

It follows Proposition 4.1 that deductible insurance is optimal insurance. We now proceed to determine the optimal deductible amount; that is, we seek the optimal l^* by solving the following optimal problem:

$$\min_{l \geq 0} \mathbb{E}\left[\exp\left\{\frac{1}{2}\left(Y + X - (X-l)_+ + (1+\theta)\int_l^\infty S_X(x)dx\right)\right\}\right]. \tag{4.12}$$

After direct calculations (See Appendix A.2 for details), we get that

$$\mathbb{E}\left[\exp\left\{\frac{1}{2}\left(Y + X - (X-l)_+ + (1+\theta)\int_l^\infty S_X(x)dx\right)\right\}\right] = e^{\frac{1}{2}(1+\theta)e^{-l}}\left[\left(4+\frac{4}{9}\alpha\right) - \left(2+\frac{2}{3}\alpha\right)e^{-\frac{1}{2}l} + \frac{2}{9}\alpha e^{-\frac{3}{2}l}\right]. \tag{4.13}$$

The numerical results for the optimal deductible l^* are given in Table 4.1. From the table, we can see that if $\theta = 0$ the optimal deductible $l^* = 0$, which is consistent with Proposition 4.2. Meanwhile, for a given α, the optimal deductible l^* increases with the increase of θ, that is, more risk will be retained if the insurance becomes more expensive. On the other hand, for a given θ, the optimal deductible l^* decrease with the increase of α; that is, when the amount of positive dependence between X and Y is increasing , the insured would cede more insurable risk to the insurer. These findings are no fluke. In fact, we can obtain the following results:

Proposition 4.7. *Let l^* be the optimal solution to the problem (4.12), then*

(i) *If $\theta = 0$, then the optimal deductible $l^* = 0$;*
(ii) *For a fixed α, the optimal deductible l^* is an increasing function of the safety loading factor θ.*
(iii) *For a fixed θ, the optimal deductible l^* is an decreasing function of the dependent parameter α.*

Note that (i) is an illustration of Proposition 4.2. The proofs of Proposition 4.7 are presented in the Appendix.

4.2. The case that $Y + X \downarrow_{st} X$

In this subsection, we investigate the optimal policy under the assumption that $Y + X \downarrow_{st} X$. This assumption is reasonable in many circumstances. For example, a person purchases a fire insurance policy for her car. The contract covers only the loss of damage to her car when a fire disaster occurs, but the loss resulting from to other causes such as an act of inappropriate use of the car, natural loss, war, military action, riot or radiation are often uninsurable. Hence, the risk of damage to her house is decomposed into an insured component X and an uninsured component Y. Because it is impossible to damage the house for two different reasons at the same time, the two components are negatively dependent in a very strong sense: whenever one is positive, the other is zero, hence they are mutually exclusive (see Chi and Liu (2017)). Hence $Y \downarrow_{st} X$. Compared to the events that corresponds to Y, the probability of the fire disaster related to X is so small that X can be somehow ignored. In such cases, it can happen that $Y + X \downarrow_{st} X$. Several other examples are also given in Dana and Scarsini (2007).

The following theorem shows that the insured will purchase no insurance under the assumption of the negative dependence.

Theorem 4.8. *If the total risk $Y + X$ is stochastically decreasing in the insurable risk X ($Y + X \downarrow_{st} X$), then purchasing no insurance is optimal.*

Proof. For any $I(x) \in \mathscr{I}$, $t \in \mathbb{R}$ and $0 \leq x_1 \leq x_2$, by applying the monotonicity of $I(x)$ and the definition of stochastically decreasing, we obtain that

$$\begin{aligned}\eta_I(x_1; t) &= \mathbb{P}(Y + X - I(X) > t \,|\, X = x_1)\\ &= \mathbb{P}(Y + X > t + I(x_1) | \, X = x_1)\\ &\geq \mathbb{P}(Y + X > t + I(x_2) | \, X = x_1)\\ &\geq \mathbb{P}(Y + X > t + I(x_2) | \, X = x_2)\\ &= \eta_I(x_2; t),\end{aligned}$$

which implies that $\eta_I(x; t)$ is decreasing. By using Corollary 3.2, the optimal insurance contract must have the following form:

$$I^*(x) \triangleq x - (x-l)_+.$$

Under insurance policy $I^*(x)$, the total risk exposure of the insured is given by

$$T_{I^*}(l) = Y + (X-l)_+ + \pi\left(\int_0^l S_X(x)dx\right).$$

Table 4.1
The optimal deductible l^* with the parameters α and θ varying.

	$\alpha=0$	$\alpha=0.2$	$\alpha=0.3$	$\alpha=0.5$	$\alpha=0.6$	$\alpha=0.8$	$\alpha=1$
$\theta=0$	0	0	0	0	0	0	0
$\theta=0.1$	0.7176	0.5973	0.5441	0.4518	0.4125	0.3464	0.2945
$\theta=0.2$	1.0493	0.9292	0.8736	0.7712	0.7243	0.6393	0.5655
$\theta=0.3$	1.3093	1.1910	1.1354	1.0310	0.9822	0.8909	0.8081
$\theta=0.4$	1.5293	1.4131	1.3580	1.2539	1.2046	1.1114	1.0251
$\theta=0.5$	1.7224	1.6080	1.5537	1.4505	1.4014	1.3080	1.2207

The expectation of $\mathbb{E}[(T_{I^*}(l)-t)_+]$ can be written as

$$\mathbb{E}[(T_{I^*}(l)-t)_+] = \int_0^l \left[\int_{\kappa(x;l,t)}^{\infty} S_{Y+X|X=x}(z)dz\right] f_X(x)dx + \int_l^{\infty}\left[\int_{\kappa(l;l,t)}^{\infty} S_{Y+X|X=x}(z)dz\right] f_X(x)dx,$$

with $\kappa(u;l,t) = t+u-\pi\left(\int_0^l S_X(x)dx\right)$.

Then, we have

$$\frac{d}{dl}\mathbb{E}[(T_{I^*}(l)-t)_+]$$
$$= S_X(l)\pi'\left(\int_0^l S_X(x)dx\right)\int_0^l S_{Y+X|X=x}(\kappa(x;l,t))f_X(x)dx$$
$$-\left(1-S_X(l)\pi'\left(\int_0^l S_X(x)dx\right)\right)\int_l^{\infty} S_{Y+X|X=x}(\kappa(l;l,t))f_X(x)dx.$$

By noticing that $S_{Y+X|X=x}(z)$ is decreasing with respect to x and using the second mean value theorem for integrals again, there exists $\zeta\in(l,\infty)$ such that

$$\frac{d}{dl}\mathbb{E}[(T_{I^*}(l)-t)_+]$$
$$= S_X(l)\pi'\left(\int_0^l S_X(x)dx\right)\int_0^l S_{Y+X|X=x}(\kappa(x;l,t))f_X(x)dx$$
$$-\left(1-S_X(l)\pi'\left(\int_0^l S_X(x)dx\right)\right)S_{Y+X|X=l}(\kappa(l;l,t))\int_l^{\zeta} f_X(x)dx$$
$$= S_X(l)\pi'\left(\int_0^l S_X(x)dx\right)\int_0^l S_{Y+X|X=x}(\kappa(x;l,t))f_X(x)dx$$
$$-\left(1-S_X(l)\pi'\left(\int_0^l S_X(x)dx\right)\right)S_{Y+X|X=l}(\kappa(l;l,t))$$
$$\times\left(F_X(\zeta)-\int_0^l f_X(x)dx\right)$$
$$= S_X(l)\pi'\left(\int_0^l S_X(x)dx\right)\int_0^l \left(S_{Y+X|X=x}(\kappa(x;l,t))\right.$$
$$\left.-S_{Y+X|X=l}(\kappa(l;l,t))\right)f_X(x)dx$$
$$+S_{Y+X|X=l}(\kappa(l;l,t))\left(F_X(l)-F_X(\zeta)+S_X(l)F_X(\zeta)\pi'\right.$$
$$\left.\times\left(\int_0^l S_X(x)dx\right)\right).$$

Furthermore, for $x\in(0,l)$, we have

$$S_{Y+X|X=x}(\kappa(x;l,t)) \ge S_{Y+X|X=l}(\kappa(x;l,t)) > S_{Y+X|X=l}(\kappa(l;l,t)),$$

and

$$F_X(l)-F_X(\zeta)+S_X(l)F_X(\zeta)\pi'\left(\int_0^l S_X(x)dx\right)$$
$$= F_X(\zeta)\left(\pi'\left(\int_0^l S_X(x)dx\right)-1\right)+F_X(l)\left(1-F_X(\zeta)\pi'\right.$$
$$\left.\times\left(\int_0^l S_X(x)dx\right)\right)$$
$$\ge F_X(\zeta)\left(\pi'\left(\int_0^l S_X(x)dx\right)-1\right)+F_X(l)\left(1-\pi'\left(\int_0^l S_X(x)dx\right)\right)$$
$$= (F_X(\zeta)-F_X(l))\left(\pi'\left(\int_0^l S_X(x)dx\right)-1\right)\ge 0,$$

which imply that $\frac{d}{dl}\mathbb{E}[(T_{I^*}(l)-t)_+] > 0$. Therefore, the minimum of $\mathbb{E}[(T_{I^*}-t)_+]$ is attained at $l=0$, as desired. □

Example 4.9. From Proposition 4.1 we know that the optimal insurance contract must have the following form:

$$I^*(x) \triangleq x-(x-l)_+.$$

when $Y+X\downarrow_{st}X$. Suppose that the insurance premium is given by $\Pi_I(X)=(1+\theta)\mathbb{E}[I(X)]$ with $0\le\theta\le 1$. Under insurance policy $I^*(x)$, the total risk exposure of the insured is given by

$$T_{I^*}(l) = Y+(X-l)_+ + (1+\theta)\int_0^l S_X(x)dx,$$

or equivalently

$$T_{I^*}(l) = Z-X+(X-l)_+ + (1+\theta)\int_0^l S_X(x)dx$$

with $Z=Y+X$.

Let

$$\rho(X)=\mathbb{E}\left[e^{\frac{X}{4}}\right].$$

One can check that ρ is a risk measure that preserves the stop-loss order.

Suppose that both Z and X follow exponential distribution and their distribution functions are given by

$$F_Z(z) = 1-e^{-\frac{1}{2}z},\ z\ge 0 \tag{4.14}$$

and

$$F_X(x) = 1-e^{-x},\ x\ge 0 \tag{4.15}$$

respectively.

Similar to Example 4.6, we assume that (X,Z) has a copula C_α that is a member of the Farlie–Gumbel–Morgenstern family of copulas, i.e.

$$C_\alpha(u,v) = uv+\alpha uv(1-u)(1-v),$$

with $\alpha\in[-1,0)$. The parameter α here can also be seen as a dependent parameter.

Then the joint distribution function and density function of (X, Z) are

$$F(x,z) = \left(1-e^{-x}\right)\left(1-e^{-\frac{1}{2}z}\right)\left(1+\alpha e^{-x}e^{-\frac{1}{2}z}\right), x \geq 0, z \geq 0 \quad (4.16)$$

and

$$f(x,z) = \frac{1}{2}e^{-x}e^{-\frac{1}{2}z} + \alpha\left(2e^{-2x}-e^{-x}\right)\left(e^{-z}-\frac{1}{2}e^{-\frac{1}{2}z}\right), \quad x \geq 0, z \geq 0 \quad (4.17)$$

respectively.

Furthermore, the conditional density function and conditional distribution function of Z given $X = x$ can be obtained as follows:

$$f_{Z|X}(z|x) = \frac{1}{2}e^{-\frac{1}{2}z} + \alpha\left(2e^{-x}-1\right)\left(e^{-z}-\frac{1}{2}e^{-\frac{1}{2}z}\right), \quad x \geq 0, z \geq 0 \quad (4.18)$$

and

$$F_{Z|X}(z|x) = 1-e^{-\frac{1}{2}z} + \alpha(2e^{-x}-1)\left(e^{-\frac{1}{2}z}-e^{-z}\right), \quad x \geq 0, z \geq 0 \quad (4.19)$$

Then, we have

$$\mathbb{P}(Z > z|X = x) = e^{-\frac{1}{2}z} - \alpha(2e^{-x}-1)\left(e^{-\frac{1}{2}z}-e^{-z}\right), \quad x \geq 0, z \geq 0 \quad (4.20)$$

and

$$\frac{\partial}{\partial x}\left(e^{-\frac{1}{2}z} - \alpha(2e^{-x}-1)\left(e^{-\frac{1}{2}z}-e^{-z}\right)\right) = 2\alpha e^{-x}\left(e^{-\frac{1}{2}z}-e^{-z}\right) \leq 0,$$

which implies that $\frac{\partial}{\partial x}\mathbb{P}(Z > z|X = x) \leq 0$ and $Z\downarrow_{st}X$ since $\alpha \leq 0$.

We now proceed to determine the optimal l. That is, we need to solve the following problem:

$$\min_{l\geq 0} \mathbb{E}[\rho(T_{I^*}(l))],$$

with

$$\mathbb{E}[\rho(T_{I^*}(l))] = \mathbb{E}\left[\exp\left\{\frac{1}{4}\left(Z - X + (X-l)_+ + (1+\theta)\int_0^l S_X(x)dx\right)\right\}\right],$$

which is a one-dimensional problem.

After simple algebras (See the Appendix A.3 for details), we can obtain that $\frac{d}{dl}\mathbb{E}[\rho(T_{I^*}(l))] > 0$ for $l > 0$ and $\frac{d}{dl}\mathbb{E}[\rho(T_{I^*}(l))] = 0$ at $l = 0$.

Hence, the minimum of $\mathbb{E}[\rho(T_{I^*}(l))]$ is attained at $l^* = 0$ and the optimal policy is $I^*(x) = 0$, i.e. purchasing no insurance is optimal policy.

Theorem 4.8 can be of great economic significance in the behavior of an insured. From the definition of stochastically decreasing, one can find that $Y + X\downarrow_{st}X$ means that the values of $Y + X$ will tend to decrease as the values of X increase. Therefore, the insurable risk will be offset by merging it with the background risk. This help to form a mechanism of self-insurance for the insured, which will lead to no incentive to buy insurance. Note that the same conclusion is obtained by Chi (2015) under the assumption of strongly negatively expectation dependent in the framework of mean–variance. However, Dana and Scarsini (2007) showed that neither no insurance nor full insurance is optimal, and that the optimal ceded loss function is decreasing over some interval. The reason is also due to the absence of incentive compatibility (see Chi, 2015).

5. Conclusions

In this paper, we study the problem of optimal insurance with background risk under a general criterion that preserves stop-loss order. According to the local monotonicity of conditional survival function, we derive the optimal contract forms in different types of interval. It is shown that the dependence structure between background risk and insurable risk plays a critical role in the optimal insurance design because the conditional survival function reflects the dependence between the two risks. We further obtain the optimal contract form explicitly under some special dependence structures. It is shown that the deductible insurance is optimal and Mossin's Theorem is still valid when background risk Y is stochastically increasing in insurable X, which generalize the corresponding results in Lu et al. (2012). Moreover, an individual will purchase no insurance when total risk $Y + X$ is stochastically decreasing in insurable risk X. However, for other dependence structures (for example, Y is stochastically decreasing in X and $Y + X$ is stochastically increasing in X), the optimal policy cannot be derived in our model. We leave the discussion of this issue to future study.

Acknowledgments

We would like to thank both reviewers for their careful reading and helpful comments that led to significant improvements of the initial draft. We are also sincerely grateful to Mrs. Ruifen Liang for her help with the grammar in this article.

Our research has been supported by the Natural Science Foundation of China (NSFC) (Nos. 71371138, 71771163) and the National Social Science Fund of China (No. 16ZDA052). The second author also acknowledges the support from the major project of the key Research Institute of Humanities and Social Science in China (No. 16JJD910001).

Appendix

A.1. Proof of Theorem 3.1

(i) For any $g_1(x; l) \in \mathcal{G}_l^{(1)}$, by making integration by parts from (3.1) and (3.3), we can obtain that

$$\mathbb{E}[g_1(X; l); X \in [a, b]] = I(b)F_X(b) - I(a)F_X(a) - \int_l^{l+I(b)-I(a)} F_X(x)dx. \quad (A.1)$$

Differentiating (A.1) with respect to l yields

$$\frac{\partial}{\partial l}\mathbb{E}[g_1(X; l); X \in [a, b]] = F_X(l) - F_X(l + I(b) - I(a)) \leq 0,$$

which implies that $\mathbb{E}[g_1(X; l); X \in [a, b]]$ is decreasing with respect to l.

We now consider the following two functions which belong to $\mathcal{G}_l^{(1)}$:

$$g_1(x; a) = I(a) + (x-a)_+ - (x - a - I(b) + I(a))_+$$

and

$$g_1(x; b - I(b) + I(a)) = I(a) + (x - b + I(b) - I(a))_+ - (x-b)_+.$$

By applying the Lipschitz continuity of $I(x)$, one can easily check that

$$g_1(x; b - I(b) + I(a)) \leq I(x) \leq g_1(x; a)$$

for $x \in [a, b)$. Thus, there exists

$$g_1^*(x) \triangleq g_1(x; l^*) = I(a) + (x-l^*)_+ - (x - l^* - I(b) + I(a))_+ \in \mathcal{G}_l^{(1)}$$

with $l^* \in [a, b - I(b) + I(a)]$ such that

$$\mathbb{E}[g_1^*(X); X \in [a,b)] = \mathbb{E}[I(X); X \in [a,b)], \tag{A.2}$$

which implies

$$\Pi_{g_1^*}(X; X \in [a,b)) = \Pi_0.$$

We now proceed to show that g_1^* satisfies (3.4), thus completing the proof of (i). That is, we need to show that

$$\begin{aligned}
&\mathbb{E}\left[(Y + X - g_1^*(X) - t)_+ | X \in [a,b)\right] \\
&\quad - \mathbb{E}\left[(Y + X - I(X) - t)_+ | X \in [a,b)\right] \\
&= \mathbb{E}[(Y + X - g_1^*(X) - t)_+ - (Y + X - I(X) - t)_+ | X \in [a,b)] \\
&\leq 0
\end{aligned}$$

for any $t \in \mathbb{R}$.

First, it follows from the Lipschitz continuity and monotonicity of $I(x)$ that there exists a certain $\lambda \in [l^*, l^* + I(b) - I(a)]$ such that $g_1^*(x) \leq I(x)$ for $x \in [a, \lambda)$ and $g_1^*(x) \geq I(x)$ for $x \in [\lambda, b)$. Then, we have

$$\begin{aligned}
&\mathbb{E}\left[(Y + X - g_1^*(X) - t)_+ - (Y + X - I(X) - t)_+ | X \in [a,b)\right] \\
&= \mathbb{E}\Big[\mathbb{E}[(Y + X - g_1^*(X) - t)_+ \\
&\quad - (Y + X - I(X) - t)_+ |X]| X \in [a,b)\Big] \\
&= \mathbb{E}\left[\int_{t-X+g_1^*(X)}^{t-X+I(X)} S_{Y|X}(y)dy | X \in [a,b)\right] \\
&\leq \mathbb{E}[(I(X) - g_1^*(X))P(Y + X - g_1^*(X) > t | X)| X \in [a,b)] \\
&= \mathbb{E}[\eta_{g_1^*}(X;t)(I(X) - g_1^*(X))| X \in [a,b)] \\
&= \int_a^\lambda \eta_{g_1^*}(x;t)(I(x) - g_1^*(x))f_X(x)dx + \int_\lambda^b \eta_{g_1^*}(x;t) \\
&\quad \times (I(x) - g_1^*(x))f_X(x)dx.
\end{aligned} \tag{A.3}$$

Since for any $t \in \mathbb{R}$, $\eta_{g_1^*}(x;t)$ is increasing in $[a,b)$ with respect to x, then by using the second mean value theorem for integrals, there exists some $\xi \in [a, \lambda]$ such that

$$\begin{aligned}
&\int_a^\lambda \eta_{g_1^*}(x;t)(I(x) - g_1^*(x))f_X(x)dx \\
&= \eta_{g_1^*}(\lambda;t)\int_\xi^\lambda (I(x) - g_1^*(x))f_X(x)dx \\
&= \eta_{g_1^*}(\lambda;t)\left(\int_a^b (I(x) - g_1^*(x))f_X(x)dx\right. \\
&\quad \left. - \int_a^\xi (I(x) - g_1^*(x))f_X(x)dx - \int_\lambda^b (I(x) - g_1^*(x))f_X(x)dx\right) \\
&= -\eta_{g_1^*}(\lambda;t)\left(\int_a^\xi (I(x) - g_1^*(x))f_X(x)dx\right. \\
&\quad \left. + \int_\lambda^b (I(x) - g_1^*(x))f_X(x)dx\right),
\end{aligned} \tag{A.4}$$

in which the equalities in the last line follows from (A.2).

Substituting (A.4) into (A.3) yields

$$\begin{aligned}
&\mathbb{E}[(Y + X - g_1^*(X) - t)_+ - (Y + X - I(X) - t)_+ | X \in [a,b)] \\
&\leq \eta_{g_1^*}(\lambda;t)\int_a^\xi (g_1^*(x) - I(x))f_X(x)dx \\
&\quad + \int_\lambda^b (\eta_{g_1^*}(x;t) - \eta_{g_1^*}(\lambda;t))(I(x) - g_1^*(x))f_X(x)dx.
\end{aligned}$$

It is clear that $\eta_{g_1^*}(\lambda;t) \geq 0$ and $\eta_{g_1^*}(x;t) - \eta_{g_1^*}(\lambda;t) \geq 0$ for $x \in [\lambda, b)$ since $\eta_{g_1^*}(x;t)$ is increasing. In addition, $g_1^*(x) - I(x) \leq 0$ for $x \in [a, \xi]$ and $I(x) - g_1^*(x) \leq 0$ for $x \in [\lambda, b)$. Then we obtain that

$$\begin{aligned}
&\mathbb{E}[(Y + X - g_1(X;l^*) - t)_+ - (Y + X - I(X) - t)_+ | X \in [a,b)] \\
&\leq 0,
\end{aligned}$$

which completes the proof of (i).

(ii) For any $h_1(x;l) \in \mathscr{H}_l^{(1)}$, it follows from (3.1) that

$$\begin{aligned}
\mathbb{E}[h_1(X;l); X \in [a,b)] &= I(b)F_X(b) - I(a)F_X(a) - \int_a^l F_X(x)dx \\
&\quad - \int_{l+b-a-I(b)+I(a)}^b F_X(x)dx.
\end{aligned}$$

and

$$\frac{\partial}{\partial l}\mathbb{E}[h_1(X;l); X \in [a,b)] = F_X(l + b - a - I(b) + I(a)) - F_X(l) \geq 0,$$

hence $\mathbb{E}[h_1(X;l); X \in [a,b)]$ is increasing with respect to l.

Let

$$h_1(x;a) = I(a) + x - a - (x-a)_+ + (x - b + I(b) - I(a))_+$$

and

$$\begin{aligned}
h_1(x; a + I(b) - I(a)) &= I(a) + x - a - (x - a - I(b) \\
&\quad + I(a))_+ + (x-b)_+.
\end{aligned}$$

It is easy to check that

$$h_1(x;a) \leq I(x) \leq h_1(x; a + I(b) - I(a))$$

for $x \in [a,b)$ and $h_1(x;a), h_1(x; a + I(b) - I(a)) \in \mathscr{H}_l^{(1)}$. Then by the continuity of $\mathbb{E}[h_1(X;l)]$, there exists

$$\begin{aligned}
h_1^* \triangleq h_1(x;l^*) &= I(a) + x - a - (x - l^*)_+ \\
&\quad + (x - l^* - b + a + I(b) - I(a))_+ \in \mathscr{H}_l^{(1)}
\end{aligned}$$

with $l^* \in [a, a + I(b) - I(a)]$ such that

$$\mathbb{E}[h_1(X;l^*); X \in [a,b)] = \mathbb{E}[I(X); X \in [a,b)], \tag{A.5}$$

and

$$\Pi_{h_1^*}(X; X \in [a,b)) = \Pi_0.$$

In addition, there exists a certain $\lambda \in [l^*, l^* + b - a - I(b) + I(a)]$ such that $h_1^*(x) \geq I(x)$ for $x \in [a, \lambda)$ and $h_1^*(x) \leq I(x)$ for $x \in [\lambda, b)$.

Since for any $t \in \mathbb{R}$, $\eta_{h_1^*}(x;t)$ is decreasing in $[a,b)$ with respect to x, by using the second mean value theorem for integrals again, there exists some $\xi \in [\lambda, b]$ such that

$$\begin{aligned}
&\int_\lambda^b \eta_{h_1^*}(x;t)(I(x) - h_1^*(x))f_X(x)dx \\
&\quad = \eta_{h_1^*}(\lambda;t)\int_\lambda^\xi (I(x) - h_1^*(x))f_X(x)dx.
\end{aligned} \tag{A.6}$$

Then similar to (A.3), we can obtain

$$\begin{aligned}
&\mathbb{E}[(Y + X - h_1^*(X) - t)_+ | X \in [a,b)] \\
&\quad - \mathbb{E}[(Y + X - I(X) - t)_+ | X \in [a,b)] \\
&\leq \int_a^\lambda \eta_{h_1^*}(x;t)(I(x) - h_1^*(x))f_X(x)dx \\
&\quad + \int_\lambda^b \eta_{h_1^*}(x;t)(I(x) - h_1^*(x))f_X(x)dx.
\end{aligned} \tag{A.7}$$

It follows from (A.5)–(A.7) that

$$\mathbb{E}[(Y+X-h_1^*(X)-t)_+|\,X\in[a,b)] - \mathbb{E}[(Y+X-I(X)-t)_+|\,X\in[a,b)] \le \int_a^{\lambda}(\eta_{h_1^*}(x;t)-\eta_{h_1^*}(\lambda;t))(I(x)-h_1^*(x))f_X(x)dx + \eta_{h_1^*}(\lambda;t)\int_{\xi}^{b}(h_1^*(x)-I(x))f_X(x)dx.$$

From the decreasing property of $\eta_{h_1^*}(x;t)$ and the relation between $h_1^*(x)$ and $I(x)$, we can conclude that

$$\mathbb{E}[(Y+X-h_1^*(X)-t)_+|\,X\in[a,b)] - \mathbb{E}[(Y+X-I(X)-t)_+|\,X\in[a,b)] \le 0,$$

as desired. □

A.2. Proof of Proposition 4.7

Let $u=e^{-\frac{1}{2}l}$. Then we have $0\le u\le 1$, $\frac{du}{dl}=-\frac{1}{2}e^{-\frac{1}{2}l}<0$ and

$$\mathbb{E}\left[\exp\left\{\frac{1}{2}\left(Y+X-(X-l)_++(1+\theta)\int_l^{\infty}S_X(x)dx\right)\right\}\right] = \gamma(u;\alpha,\theta), \tag{A.8}$$

with $\gamma(u;\alpha,\theta)\triangleq e^{\frac{1}{2}(1+\theta)u^2}\left[\left(4+\frac{4}{9}\alpha\right)-\left(2+\frac{2}{3}\alpha\right)u+\frac{2}{9}\alpha u^3\right]$.

(i) If $\theta=0$, then we have

$$\gamma(u;\alpha,0)=e^{\frac{1}{2}u^2}\left[\left(4+\frac{4}{9}\alpha\right)-\left(2+\frac{2}{3}\alpha\right)u+\frac{2}{9}\alpha u^3\right]$$

and

$$\frac{\partial}{\partial u}\gamma(u;\alpha,0)=e^{\frac{1}{2}u^2}\beta_0(u;\alpha,0)$$

with $\beta_0(u;\alpha,0)=-\left(2+\frac{2}{3}\alpha\right)+\left(4+\frac{4}{9}\alpha\right)u-2u^2+\frac{2}{9}\alpha u^4$.

By noticing the facts that $\beta_0(1;\alpha,0)=0$ and

$$\frac{\partial}{\partial u}\beta_0(u;\alpha,0)=4(1-u)+\frac{4}{9}\alpha+\frac{8}{9}\alpha u^3>0$$

for $u\in(0,1)$, one can find that $\beta_0(u;\alpha,0)<0$ and $\frac{\partial}{\partial u}\gamma(u;\alpha,0)<0$ for $u\in(0,1)$. Hence, we have

$$\frac{\partial}{\partial l}\mathbb{E}\left[\exp\left\{\frac{1}{2}\left(Y+X-(X-l)_++(1+\theta)\int_l^{\infty}S_X(x)dx\right)\right\}\right] = \frac{\partial\gamma(u;\alpha,0)}{\partial u}\frac{du}{dl}>0$$

for $l\in(0,+\infty)$, which implies that $l^*=0$ be the optimal solution to the problem (4.12) when $\theta=0$.

(ii) First, we can get that

$$\frac{\partial}{\partial u}\gamma(u;\alpha,\theta)=e^{\frac{1}{2}(1+\theta)u^2}\beta_1(u;\alpha,\theta) \tag{A.9}$$

with

$$\beta_1(u;\alpha,\theta)=-\left(2+\frac{2}{3}\alpha\right)+(1+\theta)\left(4+\frac{4}{9}\alpha\right)u - \left(2+2\theta+\frac{2}{3}\alpha\theta\right)u^2+\frac{2}{9}\alpha(1+\theta)u^4.$$

Furthermore, we have

$$\upsilon(u;\alpha,\theta)\triangleq\frac{\partial}{\partial u}\beta_1(u;\alpha,\theta)=(1+\theta)\left(4+\frac{4}{9}\alpha\right) - \left(4+4\theta+\frac{4}{3}\alpha\theta\right)u+\frac{8}{9}\alpha(1+\theta)u^3. \tag{A.10}$$

It follows from (A.10) that

$$\upsilon(0;\alpha,\theta)=(1+\theta)\left(4+\frac{4}{9}\alpha\right)>\frac{4}{3}\alpha=\upsilon(1;\alpha,\theta)\ge 0$$

and

$$\frac{\partial}{\partial u}\upsilon(u;\alpha,\theta)=-\left(4+4\theta+\frac{4}{3}\alpha\theta\right)+\frac{8}{3}\alpha(1+\theta)u^2 \le -4-4\theta+\frac{4}{3}\alpha\theta+\frac{8}{3}\alpha\le 0,$$

which implies that $\upsilon(u;\alpha,\theta)\ge 0$, or equivalently

$$\frac{\partial}{\partial u}\beta_1(u;\alpha,\theta)\ge 0 \tag{A.11}$$

for $u\in[0,1]$.

It is easy to verify that $\beta_1(0;\alpha,\theta)<0$ and $\beta_1(1;\alpha,\theta)>0$, which, together with (A.11), implies that there exists some $u_0\in(0,1)$ such that $\beta_1(u;\alpha,\theta)\le 0$ when $u\le u_0$ and $\beta_1(u;\alpha,\theta)>0$ when $u\ge u_0$. Then from the smoothness of $\gamma(u;\alpha,\theta)$, we conclude that the minimum of $\gamma(u;\alpha,\theta)$ must be attained at its stationary point. Hence, from (A.9), the optimal deductible l^* to the problem (4.12) can be obtained by solving the equations $\beta_1(u;\alpha,\theta)=0$ and $u=e^{-\frac{1}{2}l}$. Hence, we have

$$-\left(2+\frac{2}{3}\alpha\right)+(1+\theta)\left(4+\frac{4}{9}\alpha\right)u^*-\left(2+2\theta+\frac{2}{3}\alpha\theta\right)u^{*2} +\frac{2}{9}\alpha(1+\theta)u^{*4}=0. \tag{A.12}$$

with $u^*=e^{-\frac{1}{2}l^*}$.

From (A.12), for a fixed α, u^* is a function of θ, which is denoted by $u^*=u^*(\theta)$. Then for a given α, differentiating both sides of (A.12) with respect to θ yields

$$\beta_2(u^*;\alpha,\theta)\frac{du^*}{d\theta}=u^*\beta_3(u^*;\alpha,\theta) \tag{A.13}$$

with

$$\beta_2(u^*;\alpha,\theta)\triangleq(1+\theta)\left(4+\frac{4}{9}\alpha\right)-\left(4+4\theta+\frac{4}{3}\alpha\theta\right)u^* +\frac{8}{9}\alpha(1+\theta)u^{*3}$$

and

$$\beta_3(u^*;\alpha,\theta)\triangleq-\left(4+\frac{4}{9}\alpha\right)+\left(2+\frac{2}{3}\alpha\right)u^*-\frac{2}{9}\alpha u^{*3}.$$

By noticing that $\beta_2(1;\alpha,\theta)=\frac{4}{3}\alpha>0$ and

$$\frac{\partial}{\partial u^*}\beta_2(u^*;\alpha,\theta)=-4-4\theta-\frac{4}{3}\alpha\theta+\frac{8}{3}\alpha(1+\theta)u^{*2} \le -4+\frac{8}{3}\alpha+4\theta\left(\frac{\alpha}{3}-1\right)<0,$$

we get that $\beta_2(u^*;\alpha,\theta)>\beta_2(1;\alpha,\theta)>0$ for any $u^*\in[0,1)$. Similarly, one can find that $\beta_3(1;\alpha,\theta)=-2<0$ and $\frac{\partial}{\partial u^*}\beta_3(u^*;\alpha,\theta)=2+\frac{2}{3}\alpha-\frac{2}{3}\alpha u^{*2}>0$, which implies that $\beta_3(u^*;\alpha,\theta)<\beta_3(1;\alpha,\theta)<0$ for any $u^*\in[0,1)$. Then from (A.13), we have $\frac{du^*}{d\theta}<0$, which implies that $\frac{dl^*}{d\theta}=\frac{dl^*}{du^*}\frac{du^*}{d\theta}>0$.

(iii) From the proof of (ii), we know that the optimal deductible l^* to the problem (4.12) satisfies (A.12) with $u^*=e^{-\frac{1}{2}l^*}$. Similarly, for a given θ, u^* is a function of α, denoted by $u^*=u^*(\alpha)$. By taking derivative on both sides of (A.12) with respect to α, we get

$$\beta_2(u^*;\alpha,\theta)\frac{du^*}{d\alpha}=\frac{2}{9}\beta_4(u^*;\alpha,\theta) \tag{A.14}$$

with

$$\beta_4(u^*;\alpha,\theta)=3-2(1+\theta)u^*+3\theta u^{*2}-(1+\theta)u^{*4}.$$

From the proof of (ii), it suffices to prove that $\beta_4(u^*;\alpha,\theta)\geq 0$ for $u^*\in[0,1]$. After some calculations, we can obtain the following results:

$$\beta_4(0;\alpha,\theta)=3>0,\ \ \beta_4(1;\alpha,\theta)=0, \tag{A.15}$$

$$\frac{\partial}{\partial u^*}\beta_4(u^*;\alpha,\theta)=-2(1+\theta)+6\theta u^*-4(1+\theta)u^{*3},$$

$$\begin{aligned}\frac{\partial}{\partial u^*}\beta_4(u^*;\alpha,\theta)|_{u^*=0} &= -2(1+\theta)<0,\ \ \frac{\partial}{\partial u^*}\beta_4(u^*;\alpha,\theta)|_{u^*=1}\\ &= -6<0\end{aligned} \tag{A.16}$$

and

$$\frac{\partial^2}{\partial u^{*2}}\beta_4(u^*;\alpha,\theta)=6\theta-12(1+\theta)u^{*2}. \tag{A.17}$$

From (A.17), one can find that $\frac{\partial}{\partial u^*}\beta_4(u^*;\alpha,\theta)$ is increasing with respect to u^* for $u^*\in[0,u_0^*]$, while decreasing for $u^*\in(u_0^*,1]$, where $u_0^*=\sqrt{\frac{\theta}{2(1+\theta)}}$. Furthermore, we have

$$\begin{aligned}&\frac{\partial}{\partial u^*}\beta_4(u^*;\alpha,\theta)|_{u^*=u_0^*}\\ &=-2(1+\theta)+4\theta\sqrt{\frac{\theta}{2(1+\theta)}}<-2(1+\theta)+4\theta\leq 0,\end{aligned}$$

which, together with (A.16), implies that

$$\frac{\partial}{\partial u^*}\beta_4(u^*;\alpha,\theta)<0 \tag{A.18}$$

for $u^*\in[0,1]$. Now the desired result follows from (A.15) and (A.18). □

A.3. The relevant calculations in Example 4.9

First, we have

$$\begin{aligned}&\mathbb{E}[\rho(T_{I^*}(l))]\\ &=\mathbb{E}\left[\exp\left\{\frac{1}{4}\left(Z-X+(X-l)_++(1+\theta)\int_0^l S_X(x)dx\right)\right\}\right]\\ &=\exp\left\{\frac{1}{4}(1+\theta)\int_0^l S_X(x)dx\right\}\mathbb{E}\\ &\quad\times\left[\exp\left\{\frac{1}{4}\left(Z-X+(X-l)_+\right)\right\}\right]\\ &=e^{\frac{1}{4}(1+\theta)\left(1-e^{-l}\right)}\left(\int_0^l\mathbb{E}[e^{\frac{1}{4}Z}|X=x]e^{-\frac{5}{4}x}dx\right.\\ &\quad\left.+e^{-\frac{1}{4}l}\int_l^{+\infty}\mathbb{E}[e^{\frac{1}{4}Z}|X=x]e^{-x}dx\right).\end{aligned} \tag{A.19}$$

Furthermore,

$$\begin{aligned}&\mathbb{E}\left[e^{\frac{1}{4}Z}|X=x\right]\\ &=\int_0^{+\infty}e^{\frac{1}{4}z}\left(\frac{1}{2}e^{-\frac{1}{2}z}+\alpha(2e^{-x}-1)\left(e^{-z}-\frac{1}{2}e^{-\frac{1}{2}z}\right)\right)dz\\ &=\frac{1}{2}\int_0^{+\infty}e^{-\frac{1}{4}z}dz+\alpha(2e^{-x}-1)\int_0^{+\infty}\left(e^{-\frac{3}{4}z}-\frac{1}{2}e^{-\frac{1}{4}z}\right)dz\\ &=2+\frac{2}{3}\alpha-\frac{4}{3}\alpha e^{-x},\end{aligned} \tag{A.20}$$

$$\begin{aligned}&\int_0^l\mathbb{E}\left[e^{\frac{1}{4}Z}|X=x\right]e^{-\frac{5}{4}x}dx\\ &=\int_0^l\left(2+\frac{2}{3}\alpha-\frac{4}{3}\alpha e^{-x}\right)e^{-\frac{5}{4}x}dx\\ &=\frac{8}{5}-\frac{8}{135}\alpha-\left(\frac{8}{5}+\frac{8}{15}\alpha\right)e^{-\frac{5}{4}l}+\frac{16}{27}\alpha e^{-\frac{9}{4}l}\end{aligned} \tag{A.21}$$

and

$$\begin{aligned}&\int_l^{+\infty}\mathbb{E}\left[e^{\frac{1}{4}Z}|X=x\right]e^{-x}dx\\ &=\int_l^{+\infty}\left(2+\frac{2}{3}\alpha-\frac{4}{3}\alpha e^{-x}\right)e^{-x}dx\\ &=\left(2+\frac{2}{3}\alpha\right)e^{-l}-\frac{2}{3}\alpha e^{-2l}.\end{aligned} \tag{A.22}$$

Substituting (A.21) and (A.22) into (A.19) yields

$$\begin{aligned}&\mathbb{E}[\rho(T_{I^*}(l))]\\ &=e^{\frac{1}{4}(1+\theta)\left(1-e^{-l}\right)}\left\{\left(\frac{8}{5}-\frac{8}{135}\alpha\right)+\left(\frac{2}{5}+\frac{2}{15}\alpha\right)e^{-\frac{5}{4}l}\right.\\ &\quad\left.-\frac{2}{27}\alpha e^{-\frac{9}{4}l}\right\}.\end{aligned} \tag{A.23}$$

By taking the derivative of (A.23) with respect to l, we get

$$\begin{aligned}&\frac{d}{dl}\mathbb{E}[\rho(T_{I^*}(l))]\\ &=\frac{1}{4}(1+\theta)e^{-l}e^{\frac{1}{4}(1+\theta)\left(1-e^{-l}\right)}\left\{\left(\frac{8}{5}-\frac{8}{135}\alpha\right)\right.\\ &\quad\left.+\left(\frac{2}{5}+\frac{2}{15}\alpha\right)e^{-\frac{5}{4}l}-\frac{2}{27}\alpha e^{-\frac{9}{4}l}\right\}\\ &\quad+e^{\frac{1}{4}(1+\theta)\left(1-e^{-l}\right)}\left\{-\left(\frac{1}{2}+\frac{1}{6}\alpha\right)e^{-\frac{5}{4}l}+\frac{1}{6}\alpha e^{-\frac{9}{4}l}\right\}.\end{aligned} \tag{A.24}$$

We can find that $\frac{8}{5}-\frac{8}{135}\alpha>0$, $\frac{2}{5}+\frac{2}{15}\alpha>0$, $-\frac{2}{27}\alpha>0$ since $\alpha\in[-1,0)$. Hence, we have

$$\begin{aligned}\frac{d}{dl}\mathbb{E}[\rho(T_{I^*}(l))]\geq{}&\frac{1}{4}e^{-l}e^{\frac{1}{4}(1+\theta)\left(1-e^{-l}\right)}\left\{\left(\frac{8}{5}-\frac{8}{135}\alpha\right)\right.\\ &\left.+\left(\frac{2}{5}+\frac{2}{15}\alpha\right)e^{-\frac{5}{4}l}-\frac{2}{27}\alpha e^{-\frac{9}{4}l}\right\}\\ &+e^{\frac{1}{4}(1+\theta)\left(1-e^{-l}\right)}\left\{-\left(\frac{1}{2}+\frac{1}{6}\alpha\right)e^{-\frac{5}{4}l}+\frac{1}{6}\alpha e^{-\frac{9}{4}l}\right\}\\ ={}&e^{\frac{1}{4}(1+\theta)\left(1-e^{-l}\right)-l}g(l),\end{aligned}$$

where

$$\begin{aligned}g(l)={}&\left(\frac{2}{5}-\frac{2}{135}\alpha\right)-\left(\frac{1}{2}+\frac{1}{6}\alpha\right)e^{-\frac{1}{4}l}+\left(\frac{1}{10}+\frac{1}{5}\alpha\right)e^{-\frac{5}{4}l}\\ &-\frac{1}{54}\alpha e^{-\frac{9}{4}l}.\end{aligned}$$

It is easy to verify that $g(0)=0$. Furthermore, for $l>0$, we can obtain

$$\begin{aligned}\frac{dg(l)}{dl}&=\left(\frac{1}{8}+\frac{1}{24}\alpha\right)e^{-\frac{1}{4}l}-\left(\frac{1}{8}+\frac{1}{4}\alpha\right)e^{-\frac{5}{4}l}+\frac{1}{24}\alpha e^{-\frac{9}{4}l}\\ &>\left(\frac{1}{8}+\frac{1}{24}\alpha\right)e^{-\frac{5}{4}l}-\left(\frac{1}{8}+\frac{1}{4}\alpha\right)e^{-\frac{5}{4}l}+\frac{1}{24}\alpha e^{-\frac{9}{4}l}\\ &=-\frac{5}{24}\alpha e^{-\frac{5}{4}l}+\frac{1}{24}\alpha e^{-\frac{9}{4}l}\end{aligned}$$

$$> -\frac{5}{24}\alpha e^{-\frac{9}{4}l} + \frac{1}{24}\alpha e^{-\frac{9}{4}l}$$
$$= -\frac{1}{6}\alpha e^{-\frac{9}{4}l} > 0,$$

which implies that $g(l) > 0$. $\square$

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