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Self- and mutual inductances of long coaxial helical conductors

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Abstract
An analytical expression for the mutual inductance for long coaxial helical conductors or solenoids is derived on the basis of Neumann's formula for the whole range from 0 to ∞ of the pitch length, including the cases of the mutual inductance of long concentric closely wound helical solenoids and that of long parallel thin conductors as two extreme cases. In addition, an approximate expression for the self-inductance for a long helical round conductor is obtained.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

The inductance calculation of helical conductors or coils has been studied analytically and numerically, using the integral expression for the vector potential of an infinitely long helical conductor [1–3]. Then, the analytical expression for the intrinsic principal term of the mutual inductance of long coaxial thin helical conductors was obtained; however, the analytical expression for the self-inductance was not obtained [1]. Then, numerical calculation of the integral expression for inductance was used for the analysis of the current distribution of a twisted superconducting multifilamentary composite which consists of many superconducting helical conductors [4]. However, the numerical method of calculating the integral expression needed a long time for calculating the inductance matrix of the circuit equation.

In this paper, the full analytical expression for the mutual inductance of long coaxial helical thin conductors is again studied using Neumann's formula. Then, the approximate expression for the self-inductance for a long helical round conductor is studied using the summation of external and internal inductances. The external inductance of a helical conductor is calculated as approximately the mean of the mutual inductances of the filament at the inner or outer edge of the conductor and the central filament. In this paper, the terms 'helical conductor', 'helical coil' and 'solenoid' are used with the same meaning. In particular, 'helical conductor' is used to emphasize an unclosed loop, like for the straight conductor.

2. Mutual inductance of long coaxial helical thin conductors

2.1. Derivation using Neumann's formula

A multipole expansion for a helical current which is obtained by the Fourier expansion of a periodic delta function can be used with Neumann's formula. Then, the mutual inductance between two long coaxial helical thin conductors of winding radius r_1 and twist pitch length l_1 $(=2\pi/k_1)$, passing through $(r_1, \varphi_1, z = 0)$ of the circular cylindrical coordinate, r_2 and l_2 $(=2\pi/k_2)$, passing through $(r_2, \varphi_2, z = 0)$ as shown in figure 1, can be obtained using Neumann's formula, as follows:

$$L_{12} = \frac{\mu_0}{4\pi}\frac{1}{I_1 I_2}\int\int \frac{I_1 \mathrm{d}\vec{s}_1 \cdot I_2 \mathrm{d}\vec{s}_2}{|\vec{s}_1 - \vec{s}_2|} = \frac{\mu_0}{4\pi}\frac{1}{I_1 I_2} \times \int_0^{2\pi}\int_0^{l}\int_0^{2\pi}\int_0^{l}[\vec{j}_1(\theta_1, z_1) r_1 \mathrm{d}\theta_1 \mathrm{d}z_1 \cdot \vec{j}_2(\theta_2, z_2) r_2 \mathrm{d}\theta_2 \mathrm{d}z_2] \times [r_1^2 + r_2^2 + (z_1 - z_2)^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2)]^{-1/2}. \quad (1)$$

With the condition $l \gg r_2$ $(>r_1)$, the above integration can be performed analytically, like the derivation of the vector potential of an infinitely long helical current [2, 3]. As shown in figure 2, the geometrical relations between two coaxial

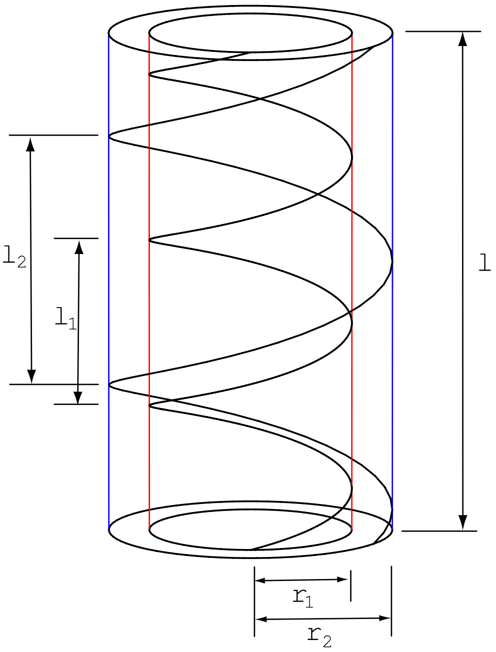

Figure 1. Two coaxial helical conductors of winding radius r_1 and pitch length l_1 (=$2\pi/k_1$), and winding radius r_2 and pitch length l_2 (=$2\pi/k_2$). In this paper, the mutual inductance of two long coaxial helical conductors is only discussed under the condition $l \gg r_2$.

cylindrical currents can be expressed as follows:

$$\begin{aligned}\hat{z}_1 &= \hat{z}_2 = \hat{z}\\ \hat{\theta}_2 &= \sin(\theta_1-\theta_2)\hat{r}_1 + \cos(\theta_1-\theta_2)\hat{\theta}_1\\ \hat{\theta}_1 &= \hat{\theta}\\ \hat{r}_1 &= \hat{r}.\end{aligned} \tag{2}$$

Then, the numerator of equation (1), namely, the term related to the current density, can be expressed as follows:

$$\vec{j}_1(\theta_1,z_1)\cdot\vec{j}_2(\theta_2,z_2) = j_z(\theta_1,z_1)\,j_z(\theta_2,z_2) + j_\theta(\theta_1,z_1)\,j_{\theta_2}(\theta_2,z_2)\cos(\theta_1-\theta_2) \tag{3}$$

where the current densities $j_z(\theta,z)$ and $j_\theta(\theta,z)$ can be expressed as follows:

$$\begin{aligned}j_z(\theta,z) &= \frac{I}{a}\sum_{m=-\infty}^{\infty}\delta(\theta-\varphi-kz-mkl_{\rm p})\\ &= \frac{I}{a}\sum_{m=-\infty}^{\infty}\delta(\theta-(\varphi+kz)-2\pi m)\\ &= \frac{I}{2\pi a}\left\{1+2\sum_{n=1}^{\infty}\cos[n(\theta-\varphi-kz)]\right\}\end{aligned} \tag{4}$$

$$\begin{aligned}j_\theta(\theta,z) &= I\sum_{m=-\infty}^{\infty}\delta\left(z-\frac{\theta-\varphi}{k}-ml_{\rm p}\right)\\ &= Ik\sum_{m=-\infty}^{\infty}\delta(\theta-\varphi-kz-2\pi m)\\ &= \frac{Ik}{2\pi}\left\{1+2\sum_{n=1}^{\infty}\cos[n(\theta-\varphi-kz)]\right\}\end{aligned} \tag{5}$$

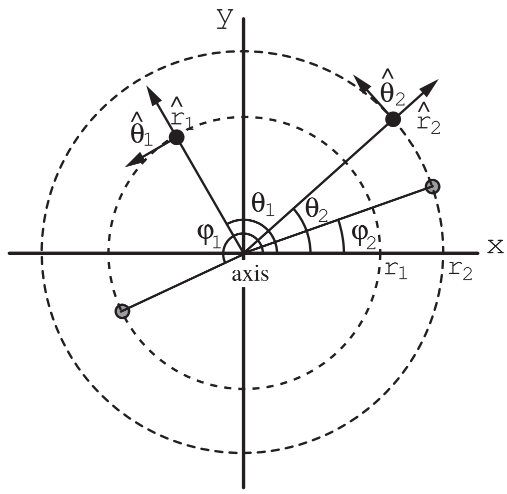

Figure 2. Cross-sectional view of two coaxial helical conductors of winding radii r_1 and r_2.

where k (=k_1 or k_2) is the twist pitch parameter. From equation (1), the mutual inductance can be calculated as follows:

$$L_{12} = L_{12,z} + L_{12,\theta} \tag{6}$$

$$\begin{aligned}L_{12,z} = \frac{\mu_0}{4\pi}\frac{1}{(2\pi)^2 r_1 r_2} &\times \int_0^{2\pi}\int_0^{l}\int_0^{2\pi}\int_0^{l}[r_1\,{\rm d}\theta_1\,{\rm d}z_1 r_2\,{\rm d}\theta_2\,{\rm d}z_2]\\ &\times[(z_1-z_2)^2+r_1^2+r_2^2-2r_1r_2\cos(\theta_1-\theta_2)]^{-1/2}\\ &\times\left\{1+2\sum_{m=1}^{\infty}\cos[m(\theta_1-\varphi_1-k_1z_1)]\right.\\ &+2\sum_{n=1}^{\infty}\cos[n(\theta_2-\varphi_2-k_2z_2)]\\ &+4\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\cos[m(\theta_1-\varphi_1-k_1z_1)]\\ &\left.\times\cos[n(\theta_2-\varphi_2-k_2z_2)]\right\}\end{aligned} \tag{7}$$

$$\begin{aligned}L_{12,\theta} = \frac{\mu_0}{4\pi}\frac{k_1k_2}{(2\pi)^2} &\times \int_0^{2\pi}\int_0^{l}\int_0^{2\pi}\int_0^{l}[\cos(\theta_1-\theta_2)r_1\,{\rm d}\theta_1\,{\rm d}z_1 r_2\,{\rm d}\theta_2\,{\rm d}z_2]\\ &\times[(z_1-z_2)^2+r_1^2+r_2^2-2r_1r_2\cos(\theta_1-\theta_2)]^{-1/2}\\ &\times\left\{1+2\sum_{m=1}^{\infty}\cos[m(\theta_1-\varphi_1-k_1z_1)]\right.\\ &+2\sum_{n=1}^{\infty}\cos[n(\theta_2-\varphi_2-k_2z_2)]\\ &+4\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\cos[m(\theta_1-\varphi_1-k_1z_1)]\\ &\left.\times\cos[n(\theta_2-\varphi_2-k_2z_2)]\right\}.\end{aligned} \tag{8}$$

By the following replacement of the integration variables and the twist parameters:

$$\begin{aligned}\theta &= \theta_1 - \theta_2\\ \theta' &= \theta_1 \quad \text{or} \quad \theta_2\\ z &= z_1 - z_2\\ z' &= z_1 + z_2\\ k_a &= \frac{k_1 + k_2}{2}\\ k_d &= \frac{k_1 - k_2}{2},\end{aligned} \tag{9}$$

the above integration for θ_1 and θ_2 can be reduced as follows:

$$\int_0^{2\pi}\int_0^{2\pi} f\left(\begin{Bmatrix}\cos\theta_1\\ \sin\theta_1\end{Bmatrix}, \begin{Bmatrix}\cos\theta_2\\ \sin\theta_2\end{Bmatrix}\right) \mathrm{d}\theta_1\,\mathrm{d}\theta_2 = \int_{\theta=0}^{\theta=2\pi}\int_{\theta'=0}^{\theta'=2\pi} g\left(\begin{Bmatrix}\cos\theta\\ \sin\theta\end{Bmatrix}, \begin{Bmatrix}\cos\theta'\\ \sin\theta'\end{Bmatrix}\right) \mathrm{d}\theta\,\mathrm{d}\theta'. \tag{10}$$

By the following simple identity:

$$\int_0^{2\pi} g'\left(\begin{Bmatrix}\cos\theta\\ \sin\theta\end{Bmatrix}\right)\mathrm{d}\theta \int_0^{2\pi}\begin{Bmatrix}\cos\theta'\\ \sin\theta'\end{Bmatrix}\mathrm{d}\theta' = 0, \tag{11}$$

the following common part of the integrand of equations (7) and (8) can be simplified with the cancellation of the term containing $\cos\theta'$ and $\sin\theta'$, as follows:

$$\begin{aligned}&1 + 2\sum_{m=1}^{\infty}\cos[m(\theta_1 - \varphi_1 - k_1 z_1)]\\ &\quad + 2\sum_{n=1}^{\infty}\cos[n(\theta_2 - \varphi_2 - k_2 z_2)]\\ &\quad + 4\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\cos[m(\theta_1 - \varphi_1 - k_1 z_1)]\\ &\quad \times \cos[n(\theta_2 - \varphi_2 - k_2 z_2)]\\ &\quad \to 1 + 2\sum_{n=1}^{\infty}\Big\{\cos[n\{(\varphi_1 - \varphi_2) + k_d z'\}]\\ &\quad \times\{\cos(n\theta)\cos(nk_a z) + \sin(n\theta)\sin(nk_a z)\}\\ &\quad + \sin[n\{(\varphi_1 - \varphi_2) + k_d z'\}]\{\sin(n\theta)\cos(nk_a z)\\ &\quad - \cos(n\theta)\sin(nk_a z)\}\Big\}.\end{aligned} \tag{12}$$

As a result, roughly, the second term has a non-zero value only for $k_1 = k_2$ ($=k$), as follows:

$$\begin{aligned}&\to 1 + 2\delta(k_1, k_2)\sum_{n=1}^{\infty}\{\cos[n(\varphi_1 - \varphi_2)]\{\cos(n\theta)\cos(nkz)\\ &\quad + \sin(n\theta)\sin(nkz)\} + \sin[n(\varphi_1 - \varphi_2)]\\ &\quad \times\{\sin(n\theta)\cos(nkz) - \cos(n\theta)\sin(nkz)\}\},\\ &\delta(k_1, k_2) \approx \begin{cases}1 & (k_1 = k_2)\\ 0 & (k_1 \neq k_2).\end{cases}\end{aligned} \tag{13}$$

In addition, the above integration for z_1 and z_2 can be reduced as follows:

$$\begin{aligned}&\int_0^l\int_0^l f(z_1, z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2\\ &= \int_{z=-l}^{z=l}\int_{z'=|z|}^{z'=2l-|z|} g(z, z')\frac{\partial(z_1, z_2)}{\partial(z, z')}\,\mathrm{d}z\,\mathrm{d}z'\\ &= \tfrac{1}{2}\int_{z=0}^{z=l}\int_{z'=z}^{z'=2l-z} g(z, z')\,\mathrm{d}z\,\mathrm{d}z'\\ &\quad + \tfrac{1}{2}\int_{z=-l}^{z=0}\int_{z'=-z}^{z'=2l+z} g(z, z')\,\mathrm{d}z\,\mathrm{d}z'\\ &= \tfrac{1}{2}\int_{z=0}^{z=l}\int_{z'=z}^{z'=2l-z} g(z, z')\,\mathrm{d}z\,\mathrm{d}z'\\ &\quad + \tfrac{1}{2}\int_{z=0}^{z=l}\int_{z'=z}^{z'=2l-z} g(-z, z')\,\mathrm{d}z\,\mathrm{d}z'\\ &= \tfrac{1}{2}\int_{z=0}^{z=l}\int_{z'=z}^{z'=2l-z}\{g(z, z') + g(-z, z')\}\,\mathrm{d}z\,\mathrm{d}z'.\end{aligned} \tag{14}$$

Furthermore, as the term containing $\sin(n\theta)$ in equation (12) cancels out, as shown below, equations (7) and (8) can be reduced as follows:

$$\begin{aligned}L_{12,z} &= \frac{\mu_0}{2(2\pi)^3}\\ &\quad \times \int_0^{2\pi}\int_0^{2\pi}\int_0^l\int_z^{2l-z}\frac{\mathrm{d}\theta\mathrm{d}\theta'\mathrm{d}z\mathrm{d}z'}{\sqrt{r_1^2 + r_2^2 + z^2 - 2r_1 r_2\cos\theta}}\\ &\quad \times\left\{1 + 2\delta(k_1, k_2)\sum_{n=1}^{\infty}\cos[n(\varphi_1 - \varphi_2)]\cos(n\theta)\cos(nkz)\right\}\end{aligned} \tag{15}$$

$$\begin{aligned}L_{12,\theta} &= \frac{\mu_0 k_1 k_2 r_1 r_2}{2(2\pi)^3}\\ &\quad \times \int_0^{2\pi}\int_0^{2\pi}\int_0^l\int_z^{2l-z}\frac{\cos\theta\mathrm{d}\theta\mathrm{d}\theta'\mathrm{d}z\mathrm{d}z'}{\sqrt{r_1^2 + r_2^2 + z^2 - 2r_1 r_2\cos\theta}}\\ &\quad \times\left\{1 + 2\delta(k_1, k_2)\sum_{n=1}^{\infty}\cos[n(\varphi_1 - \varphi_2)]\right.\\ &\quad \left.\times\cos(n\theta)\cos(nkz)\right\}.\end{aligned} \tag{16}$$

The first term of equation (15) can be calculated on the condition that $l \gg r_2$, as follows:

$$\begin{aligned}&\frac{\mu_0}{2(2\pi)^3}\int_0^{2\pi}\int_0^{2\pi}\int_{z=0}^{z=l}\int_{z'=z}^{z'=2l-z}\frac{\mathrm{d}\theta\mathrm{d}\theta'\mathrm{d}z\mathrm{d}z'}{\sqrt{z^2+r_1^2+r_2^2-2r_1 r_2\cos\theta}}\\ &= \frac{\mu_0}{2(2\pi)^2}\int_0^{2\pi}\int_0^l\frac{2(l - z)\,\mathrm{d}\theta\mathrm{d}z}{\sqrt{z^2 + r_1^2 + r_2^2 - 2r_1 r_2\cos\theta}}\\ &\approx \frac{\mu_0 l}{(2\pi)^2}\int_0^{2\pi}\left\{\ln 2l - \ln\sqrt{r_1^2+r_2^2-2r_1 r_2\cos\theta} - 1\right\}\mathrm{d}\theta\\ &= \frac{\mu_0 l}{2\pi}(\ln 2l - 1)\frac{\mu_0 l}{(2\pi)^2}\int_0^{2\pi}\left\{\ln r_2 - \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r_1}{r_2}\right)^n\right.\\ &\quad \left.\times\cos n\theta\right\}\mathrm{d}\theta = \frac{\mu_0 l}{2\pi}\left(\ln\frac{2l}{r_2} - 1\right).\end{aligned} \tag{17}$$

The first term of equation (16) can be calculated on the condition that $l \gg r_2$, as follows:

$$\begin{aligned}&\frac{\mu_0 k_1 k_2 r_1 r_2}{2(2\pi)^3}\\ &\quad \times\int_0^{2\pi}\int_0^{2\pi}\int_{z=0}^{z=l}\int_{z'=z}^{z'=2l-z}\frac{\cos\theta\mathrm{d}\theta\mathrm{d}\theta'\mathrm{d}z\mathrm{d}z'}{\sqrt{z^2 + r_1^2 + r_2^2 - 2r_1 r_2\cos\theta}}\end{aligned}$$

$$\approx \frac{\mu_0 k_1 k_2 r_1 r_2 l}{(2\pi)^2}\int_0^{2\pi}\left\{\ln 2l - \ln\sqrt{r_1^2+r_2^2-2r_1r_2\cos\theta}-1\right\}$$
$$\times\cos\theta\,\mathrm{d}\theta = -\frac{\mu_0 k_1 k_2 r_1 r_2 l}{(2\pi)^2}$$
$$\times\int_0^{2\pi}\left\{\ln r_2 - \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r_1}{r_2}\right)^n\cos n\theta\right\}\cos\,\mathrm{d}\theta$$
$$= \frac{\mu_0 k_1 k_2 r_1 r_2 l}{(2\pi)^2}\int_0^{2\pi}\left\{\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r_1}{r_2}\right)^n\right.$$
$$\times\left.\left\{\frac{\cos\{(n+1)\theta\}+\cos\{(n-1)\theta\}}{2}\right\}\right\}\mathrm{d}\theta = \frac{\mu_0 l}{4\pi}k_1k_2r_1^2. \quad (18)$$

The second term of equation (15) with $k_1 = k_2$ $(=k)$, can be calculated as follows:

$$\frac{\mu_0}{2(2\pi)^3}$$
$$\times\int_0^{2\pi}\int_0^{2\pi}\int_{z=0}^{z=l}\int_{z'=z}^{z'=2l-z}\frac{\cos(n\theta)\cos(nkz)\,\mathrm{d}\theta\,\mathrm{d}\theta'\mathrm{d}z\mathrm{d}z'}{\sqrt{r_1^2+r_2^2+z^2-2r_1r_2\cos\theta}}$$
$$= \frac{\mu_0}{(2\pi)^2}\int_0^{2\pi}\int_0^{l}\frac{\cos(n\theta)\cos(nkz)(l-z)\,\mathrm{d}\theta\mathrm{d}z}{\sqrt{r_1^2+r_2^2+z^2-2r_1r_2\cos\theta}}$$
$$\approx\frac{\mu_0 l}{(2\pi)^2}\int_0^{2\pi}K_0\left(nk\sqrt{r_1^2+r_2^2-2r_1r_2\cos\theta}\right)$$
$$\times\cos(n\theta)\mathrm{d}\theta = \frac{\mu_0 l}{(2\pi)^2}\int_0^{2\pi}\left\{\sum_{m=-\infty}^{\infty}I_m(nkr_1)K_m(nkr_2)\right.$$
$$\times\left.\cos(m\theta)\right\}\cos(n\theta)\,\mathrm{d}\theta = \frac{\mu_0 l}{(2\pi)^2}2\pi I_n(nkr_1)K_n(nkr_2)$$
$$= \frac{\mu_0 l}{2\pi}I_n(nkr_1)K_n(nkr_2). \quad (19)$$

Similarly, the second term of equation (16) with $k_1 = k_2$ $(=k)$, can be calculated as follows:

$$\frac{\mu_0 k^2 r_1 r_2}{2(2\pi)^3}$$
$$\times\int_0^{2\pi}\int_0^{2\pi}\int_{z=0}^{z=l}\int_{z'=z}^{z'=2l-z}\left[\cos(n\theta)\cos(\theta)\right.$$
$$\times\left.\cos(nkz)\,\mathrm{d}\theta\mathrm{d}\theta'\mathrm{d}z\mathrm{d}z'\right]\left[r_1^2+r_2^2+z^2-2r_1r_2\cos\theta\right]^{-1/2}$$
$$\approx\frac{\mu_0 l}{4\pi^2}k^2r_1r_2\int_0^{2\pi}K_0\left(nk\sqrt{r_1^2+r_2^2-2r_1r_2\cos\theta}\right)$$
$$\times\cos(n\theta)\cos\theta\,\mathrm{d}\theta = \frac{\mu_0 l}{4\pi^2}k^2r_1r_2$$
$$\times\int_0^{2\pi}\left\{\sum_{m=-\infty}^{\infty}I_m(nkr_1)K_m(nkr_2)\cos(m\theta)\right\}$$
$$\times\left\{\frac{\cos(n+1)\theta+\cos(n-1)\theta}{2}\,\mathrm{d}\theta\right\}$$
$$= \frac{\mu_0 l}{4\pi}k^2r_1r_2\{I_{n+1}(nkr_1)K_{n+1}(nkr_2)$$
$$+ I_{n-1}(nkr_1)K_{n-1}(nkr_2)\}. \quad (20)$$

It is shown that the term containing $\sin(n\theta)$ in equation (12) cancels out, through the replacement $\cos(n\theta)$ by $\sin(n\theta)$ in equations (19) and (20). In the mathematical manipulation of equations (19) and (20), the following relation is used [5, 6]:

$$\int_0^{\infty}\frac{\cos(\alpha x)\mathrm{d}x}{\sqrt{\beta^2+x^2}} = K_0(\alpha\beta) \qquad \alpha>0, \quad \mathrm{Re}\,\beta>0. \quad (21)$$

The addition theorem for Bessel functions is also used [6]:

$$K_0\left(nk\sqrt{r_1^2+r_2^2-2r_1r_2\cos\theta}\right) = \sum_{m=-\infty}^{\infty}I_m(nkr_1)K_m(nkr_2)$$
$$\times\cos(m\theta). \quad (22)$$

Furthermore, the condition $l \gg r_2$ is used as follows:

$$\int_{z=0}^{z=l}\frac{\cos(nkz)(l-z)\mathrm{d}z}{\sqrt{r_1^2+r_2^2+z^2-2r_1r_2\cos\theta}}$$
$$= l\int_{z=0}^{z=l}\frac{\cos(nkz)\mathrm{d}z}{\sqrt{r_1^2+r_2^2+z^2-2r_1r_2\cos\theta}}$$
$$-\int_{z=0}^{z=l}\frac{\cos(nkz)z\mathrm{d}z}{\sqrt{r_1^2+r_2^2+z^2-2r_1r_2\cos\theta}}$$
$$\approx l\int_{z=0}^{z=\infty}\frac{\cos(nkz)\mathrm{d}z}{\sqrt{r_1^2+r_2^2+z^2-2r_1r_2\cos\theta}}$$
$$-\int_{z=0}^{z=l}\cos(nkz)\,\mathrm{d}z$$
$$= lK_0\left(nk\sqrt{r_1^2+r_2^2-2r_1r_2\cos\theta}\right). \quad (23)$$

As a result, the analytical expression for the mutual inductance is obtained as follows:

$$L_{12} = \frac{\mu_0 l}{2\pi}\left(\ln\frac{2l}{r_2}-1\right)+\delta(k_1,k_2)\frac{\mu_0 l}{\pi}\sum_{n=1}^{\infty}I_n(nkr_1)K_n(nkr_2)$$
$$\times\cos[n(\varphi_2-\varphi_1)] + \frac{\mu_0 l}{4\pi}k_1k_2r_1^2 + \delta(k_1,k_2)\frac{\mu_0 l}{2\pi}k^2r_1r_2$$
$$\times\sum_{n=1}^{\infty}\{I_{n+1}(nkr_1)K_{n+1}(nkr_2)+I_{n-1}(nkr_1)K_{n-1}(nkr_2)\}$$
$$\times\cos[n(\varphi_2-\varphi_1)]. \quad (24)$$

As two extreme cases of $k_1, k_2 \to \infty$ and $k_1, k_2 \to 0$ (or $k \to 0$) of equation (24), two limiting extreme mutual inductances for two parallel straight thin conductors and two solenoids can be obtained as follows:

$$\lim_{k_1,k_2\to 0}L_{12} = \frac{\mu_0 l}{2\pi}\left(\ln\frac{2l}{r_2}-1\right)+\frac{\mu_0 l}{\pi}\sum_{n=1}^{\infty}\{\lim_{k\to 0}I_n(nkr_1)$$
$$\times K_n(nkr_2)\}\cos[n(\varphi_2-\varphi_1)] = \frac{\mu_0 l}{2\pi}(\ln 2l-1)-\frac{\mu_0 l}{2\pi}$$
$$\times\ln r_2 + \frac{\mu_0 l}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n}\left(\frac{r_1}{r_2}\right)^n\cos[n(\varphi_2-\varphi_1)]$$
$$= \frac{\mu_0 l}{2\pi}\left(\ln\frac{2l}{\sqrt{r_1^2+r_2^2-2r_1r_2\cos(\varphi_2-\varphi_1)}}-1\right)$$
$$= \frac{\mu_0 l}{2\pi}\left(\ln\frac{2l}{r_{12}}-1\right) \quad (25)$$

$$\lim_{k_1,k_2\to\infty} L_{12} = \frac{\mu_0 l}{2\pi}\left(\ln\frac{2l}{r_2}-1\right)+\frac{\mu_0 l}{4\pi}k_1k_2r_1^2$$
$$= \frac{\mu_0 l}{2\pi}\left(\ln\frac{2l}{r_2}-1\right)+\mu_0 n_1 n_2 \pi r_1^2 l \tag{26}$$

where r_{12} is the distance between two points, $(r_1, \varphi_1, z = 0)$ and $(r_2, \varphi_2, z = 0)$ in cylindrical coordinates, and n_1 $(=1/l_1)$ and n_2 $(=1/l_2)$ are the turn numbers per unit length of the inner and outer solenoids, respectively. In the mathematical manipulation of equations (25) and (26), the following asymptotic forms of the modified Bessel functions are used [7]:

$$\lim_{k\to 0} I_n(nkr_1)K_n(nkr_2) = \frac{1}{n!}\left(\frac{nkr_1}{2}\right)^n\frac{(n-1)!}{2}\left(\frac{2}{nkr_2}\right)^n$$
$$= \frac{1}{2n}\left(\frac{r_1}{r_2}\right)^n \tag{27}$$

$$\lim_{k\to\infty} I_n(nkr_1)K_n(nkr_2) = \lim_{k\to\infty}\frac{1}{\sqrt{2\pi nkr_1}}\mathrm{e}^{nkr_1}\sqrt{\frac{\pi}{2nkr_2}}\mathrm{e}^{-nkr_2}$$
$$= \lim_{k\to\infty}\frac{\pi}{2nk\sqrt{r_1r_2}}\mathrm{e}^{-nk(r_2-r_1)} = 0. \tag{28}$$

2.2. Derivation using the vector potential of an infinitely long helical thin conductor

The mutual inductance of long coaxial helical conductors has been discussed, using the vector potential of an infinitely long helical thin conductor [1]. Generally, the mutual inductance L_{12} for two parallel long straight conductors of length l with arbitrary conductor boundary can be expressed, using the geometrical mean distance (g.m.d.) R_{12}, as follows [8–10]:

$$L_{12} = \frac{\mu_0 l}{2\pi}\left(\ln\frac{2l}{R_{12}}-1\right) = \frac{\mu_0 l}{2\pi}(\ln 2l - 1) - \frac{\mu_0 l}{2\pi}\ln R_{12} \tag{29}$$

$$\ln R_{12} = \frac{1}{S_1S_2}\int\int \ln r_{12}\,\mathrm{d}S_1\,\mathrm{d}S_2 \tag{30}$$

$$L_{\mathrm{com}} = \frac{\mu_0 l}{2\pi}(\ln 2l - 1) \tag{31}$$

$$L_{12,\mathrm{r}} = -\frac{\mu_0 l}{2\pi}\ln R_{12} \tag{32}$$

where S_1 and S_2 are the cross-sectional areas of each of the conductors and r_{12} is the distance between two parallel filaments within each conductor.

Then, we can formally decompose the inductance L_{12} into two terms, i.e. the common term L_{com} and the intrinsic principal term $L_{12,\mathrm{r}}$, related to the g.m.d. The subscript r is used in equation (32), as the intrinsic principal term has been called 'the reduced inductance' [1]. This formal decomposition can also be applied for a helical conductor, as shown in equation (24). As equations (31) and (32) do not satisfy the requirement that the argument of the logarithm should be dimensionless, they should be thought of as formal expressions. However, equation (30) originally presented by Maxwell has been widely used as the defining equation for the g.m.d. [8–10].

The vector potential due to an infinitely long straight conductor parallel to the z axis passing through some point $(r = a, \theta = \varphi)$ at the $z = 0$ plane can be expressed in circular cylindrical coordinates as follows:

$$A_z(r,\theta,z) = -\frac{\mu_0 I}{2\pi}\ln\frac{\sqrt{a^2+r^2-2ar\cos(\theta-\varphi)}}{c}$$
$$= -\frac{\mu_0 I}{2\pi}\ln\frac{r_{12}}{c} = -\frac{\mu_0 I}{2\pi}\ln r_{12} + \frac{\mu_0 I}{2\pi}\ln c \tag{33}$$

where r_{12} is the distance between (a, φ) and (r, θ), and c is an arbitrary constant with the dimension of length, needed to make the argument of the logarithm dimensionless. Furthermore, for a helical thin conductor carrying the current I, with a pitch length l_{p} $(=2\pi/k)$, passing through some point $(r = a, \theta = \varphi)$ at the $z = 0$ plane, the vector potential at the general point (r, θ, z) can be expressed with an arbitrary constant c in circular cylindrical coordinates as follows [2, 3]:

for $r \leqslant a$,

$$\begin{aligned} A_r(r,\theta,z) &= -\frac{\mu_0 I}{2\pi}ka\sum_{n=1}^{\infty}\{K_{n+1}(nka)I_{n+1}(nkr) \\ &\quad - K_{n-1}(nka)I_{n-1}(nkr)\}\sin[n(\theta-\varphi-kz)] \\ A_\theta(r,\theta,z) &= \frac{\mu_0 I}{4\pi}kr + \frac{\mu_0 I}{2\pi}ka\sum_{n=1}^{\infty}\{K_{n+1}(nka) \\ &\quad \times I_{n+1}(nkr) + K_{n-1}(nka)I_{n-1}(nkr)\} \\ &\quad \times\cos[n(\theta-\varphi-kz)] \\ A_z(r,\theta,z) &= -\frac{\mu_0 I}{2\pi}\ln\frac{a}{c} + \frac{\mu_0 I}{\pi}\sum_{n=1}^{\infty}K_n(nka)I_n(nkr) \\ &\quad \times\cos[n(\theta-\varphi-kz)] \end{aligned} \tag{34}$$

and for $r \geqslant a$,

$$\begin{aligned} A_r(r,\theta,z) &= -\frac{\mu_0 I}{2\pi}ka\sum_{n=1}^{\infty}\{I_{n+1}(nka)K_{n+1}(nkr) \\ &\quad - I_{n-1}(nka)K_{n-1}(nkr)\}\sin[n(\theta-\varphi-kz)] \\ A_\theta(r,\theta,z) &= \frac{\mu_0 I}{4\pi}k\frac{a^2}{r} + \frac{\mu_0 I}{2\pi}ka\sum_{n=1}^{\infty}\{I_{n+1}(nka) \\ &\quad \times K_{n+1}(nkr) + I_{n-1}(nka)K_{n-1}(nkr)\} \\ &\quad \times\cos[n(\theta-\varphi-kz)] \\ A_z(r,\theta,z) &= -\frac{\mu_0 I}{2\pi}\ln\frac{r}{c} + \frac{\mu_0 I}{\pi}\sum_{n=1}^{\infty}I_n(nka)K_n(nkr) \\ &\quad \times\cos[n(\theta-\varphi-kz)]. \end{aligned} \tag{35}$$

The asymptotic form of the vector potential, equations (34) and (35), for a helical thin conductor coincides with equation (33) for a straight thin conductor on $k \to 0$ [1].

Then, it can be thought that the intrinsic principal term $L_{12,\mathrm{st,r}}$ of the mutual inductance of two long straight conductors at distance d can be formally obtained from the vector potential without an arbitrary constant c for an infinitely long straight conductor from equations (29) and (30), as follows:

$$\frac{L_{12,\mathrm{st,r}}}{l} = -\frac{\mu_0}{2\pi l}\int_0^l \ln r_{12}\,\mathrm{d}z = -\frac{\mu_0}{2\pi}\ln r_{12}. \tag{36}$$

As an extension, it is expected that the intrinsic principal term $L_{12,\mathrm{r}}$ of the mutual inductance of two long coaxial helical conductors of winding radius r_1 and twist pitch parameter k_1 $(=2\pi/l_1)$, and winding radius r_2 and twist pitch parameter k_2 $(=2\pi/l_2)$, as shown in figure 1, can be expressed as follows [1]:

$$\frac{L_{12,\mathrm{r}}}{l} = \frac{1}{lI}\left\{\int_0^l A_z(r_2, \varphi_2 + k_2 z, z)\,\mathrm{d}z + \int_0^{k_2 l} A_\theta\left(r_2, \varphi_2 + \theta, \frac{\theta}{k_2}\right) r_2\,\mathrm{d}\theta\right\} \quad (37)$$

where $A_z(r, \theta, z)$ and $A_\theta(r, \theta, z)$ are the vector potential without an arbitrary constant c for an infinitely long helical conductor passing through $(r_1, \varphi_1, z = 0)$. The twist pitch length l_2 of equation (32) in [1] is correctly replaced by the whole length l in equation (37). Then, the intrinsic principal term of the mutual inductance of two long helical conductors can be obtained as follows:

$$\begin{aligned}\frac{L_{12,\mathrm{r}}}{l} &= -\frac{\mu_0}{2\pi}\ln r_2 + \frac{\mu_0}{\pi}\sum_{n=1}^{\infty} I_n(nk_1r_1)K_n(nk_1r_2) \\ &\quad\times \langle\cos[n(\varphi_2-\varphi_1)]\rangle_{\mathrm{av}} + \frac{\mu_0}{4\pi}k_1k_2r_1^2 + \frac{\mu_0}{2\pi}k_1k_2r_1r_2 \\ &\quad\times \sum_{n=1}^{\infty}\big\{I_{n+1}(nk_1r_1)K_{n+1}(nk_1r_2) + I_{n-1}(nk_1r_1) \\ &\quad\times K_{n-1}(nk_1r_2)\big\}\langle\cos[n(\varphi_2-\varphi_1)]\rangle_{\mathrm{av}}\end{aligned} \quad (38)$$

where

$$\begin{aligned}&\langle\cos[n(\varphi_2-\varphi_1)]\rangle_{\mathrm{av}} \\ &\quad= \frac{1}{l}\int_0^l \cos[n\{(\varphi_2-\varphi_1) + (k_2-k_1)z\}]\,\mathrm{d}z \\ &\quad= \frac{1}{k_2 l}\int_0^{k_2 l}\cos\left[n\left\{(\varphi_2-\varphi_1) + \frac{k_2-k_1}{k_2}\theta\right\}\right]\mathrm{d}\theta \\ &\quad= \delta(k_1,k_2,\varphi_1,\varphi_2,n,l)\cos[n(\varphi_2-\varphi_1)]\end{aligned}$$

$$\begin{aligned}\delta(k_1,k_2,\varphi_1,\varphi_2,n,l) &= \frac{\sin[n(k_2-k_1)l]}{n(k_2-k_1)l} \\ &\quad+ \frac{\sin[n(\varphi_2-\varphi_1)]}{\cos[n(\varphi_2-\varphi_1)]}\frac{(\cos[n(k_2-k_1)l]-1)}{n(k_2-k_1)l} \\ &\lim_{(k_2-k_1)\to 0}\delta(k_1,k_2,\varphi_1,\varphi_2,n,l) = 1 \\ &\lim_{(k_2-k_1)l\to\infty}\delta(k_1,k_2,\varphi_1,\varphi_2,n,l) = 0.\end{aligned} \quad (39)$$

As a result, it is shown that equation (38) is identical to equation (24), except for the common term.

3. Rigorous calculation of the self-inductance

3.1. Mutual inductance of long helical conductors

The mutual inductance $L_{12,\mathrm{S}}$ of two coaxial helical thin conductors of finite cross-section with uniform helical current density can be expressed using a double surface integral, as follows:

$$L_{12,\mathrm{S}} = \frac{1}{S_1S_2}\int_{S_1}\int_{S_2} L_{12}\,\mathrm{d}S_1\,\mathrm{d}S_2 \quad (40)$$

where S_1 and S_2 are the cross-sectional areas of each of the conductors. In this paper, the self-inductance of a long helical conductor is rigorously or approximately calculated, using the above expression. In this paper, only a helical conductor with uniform helical current density is discussed. In addition, except for the common term, the integral expression for equation (40) is equivalent to equations (30) and (37) of [1].

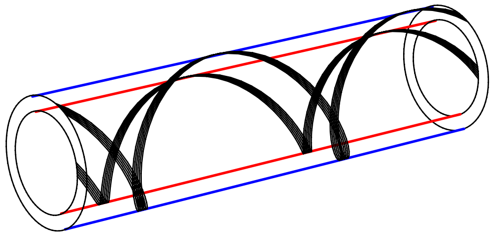

Figure 3. Two coaxial helical tape conductors with thin arcuate cross-sections.

3.2. Self-inductance of a long helical thin tape conductor

For a special case of helical tape conductors with thin arcuate cross-section, as shown in figure 3, the self-inductance and mutual inductance can be rigorously obtained, as shown below. This result is useful for the inductance calculation for a superconducting power cable which consists of tape conductors wound on a circular cylinder. The mutual inductance $L_{12,\mathrm{arc}}$ of two coaxial helical tape conductors with thin arcuate cross-section with the same pitch length l_{p} $(=2\pi/k)$ and the same axial length l, located at $(r_1, \varphi_{\mathrm{a}} \pm \Delta\varphi/2z = 0)$ and $(r_2, \varphi_b \pm \Delta\varphi/2z = 0)$, can be easily calculated from equations (24) and (40) for $r_1 < r_2$ as follows:

$$\begin{aligned}L_{12,\mathrm{arc}} &= \frac{1}{r_1\Delta\varphi_1 r_2\Delta\varphi_2} \\ &\quad\times\int_{\varphi_1=\varphi_a-\Delta\varphi_1/2}^{\varphi_1=\varphi_a+\Delta\varphi_1/2}\times\int_{\varphi_2=\varphi_b-\Delta\varphi_2/2}^{\varphi_2=\varphi_b+\Delta\varphi_2/2} L_{12}r_1\,\mathrm{d}\varphi_1 r_2\,\mathrm{d}\varphi_2 \\ &= \frac{\mu_0 l}{2\pi}\left(\ln\frac{2l}{r_2}-1\right) + \frac{\mu_0 l}{\pi\Delta\varphi_1\Delta\varphi_2}\sum_{n=1}^{\infty}\Big\{I_n(nkr_1)K_n(nkr_2) \\ &\quad\times\frac{4}{n^2}\cos[n(\varphi_b-\varphi_a)]\sin(n\Delta\varphi_1)\sin(n\Delta\varphi_2)\Big\} \\ &\quad+\frac{\mu_0 l}{4\pi}k^2r_1^2 + \frac{\mu_0 l}{2\pi\Delta\varphi_1\Delta\varphi_2}k^2r_1r_2 \\ &\quad\times\sum_{n=1}^{\infty}\{\{I_{n+1}(nkr_1)K_{n+1}(nkr_2) + I_{n-1}(nkr_1)K_{n-1}(nkr_2)\} \\ &\quad\times\frac{4}{n^2}\cos[n(\varphi_b-\varphi_a)]\sin(n\Delta\varphi_1)\sin(n\Delta\varphi_2)\}.\end{aligned} \quad (41)$$

In particular, the self-inductance $L_{11,\mathrm{arc}}$ of a helical tape conductor with a thin arcuate cross-section is expressed as follows:

$$\begin{aligned}L_{11,\mathrm{arc}} &= \frac{\mu_0 l}{2\pi}\left(\ln\frac{2l}{r_1}-1\right) + \frac{\mu_0 l}{\pi(\Delta\varphi_1)^2}\sum_{n=1}^{\infty} I_n(nkr_1)K_n(nkr_1) \\ &\quad\times\frac{4}{n^2}\sin^2(n\Delta\varphi_1) + \frac{\mu_0 l}{4\pi}k^2r_1^2 + \frac{\mu_0 l}{2\pi(\Delta\varphi_1)^2}k^2r_1^2 \\ &\quad\times\sum_{n=1}^{\infty}\{I_{n+1}(nkr_1)K_{n+1}(nkr_1) + I_{n-1}(nkr_1)K_{n-1}(nkr_1)\} \\ &\quad\times\frac{4}{n^2}\sin^2(n\Delta\varphi_1).\end{aligned} \quad (42)$$

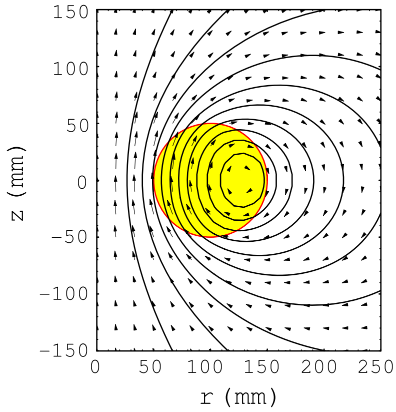

Figure 4. Contour plot of $r \times A_\theta(r, z)$ and the vector plot of the magnetic field due to the annular ring of a circular conductor with ring radius $r = 100$ mm and conductor radius $a = 50$ mm.

Then, it is confirmed that the above self-inductance with $\Delta\varphi_1 = 2\pi$, is equal to the well-known expression for a closely wound solenoid, as shown in equation (26) with $r_1 = r_2$ and $n_1 = n_2$.

4. Approximate calculation of the self-inductance

4.1. Self-inductance of an annular ring

The approximate expression for the self-inductance of an annular ring has been investigated since Kirchhoff and Maxwell, using the analytical expression for the mutual inductance of two concentric circular rings [8, 9, 11]. One simple approximate method is to calculate the self-inductances as the sum of the external and internal inductances. A helical conductor is geometrically intermediate between a straight conductor ($k = 0$) and a stack of annular rings ($k = \infty$). Therefore, if an approximate method is valid for both a straight conductor and an annular ring, it is expected that this method is also valid for a helical conductor. The self-inductance L of a long straight round conductor is rigorously expressed as the sum of the external and internal inductances [8–11].

The magnetic energy of the annular ring of a circular conductor can be expressed as follows:

$$E = \tfrac{1}{2}LI^2 = \tfrac{1}{2}\int_{S_\theta} j_\theta A_\theta(r, z) r \,\mathrm{d}r\,\mathrm{d}z \tag{43}$$

where $A_\theta(r, z)$ is the azimuthal vector potential in the circular cylindrical coordinate system at radius r and the axial distance z from the $z = 0$ plane due to a filamentary circle at the $z = 0$ plane with radius a. The azimuthal vector potential and the related modulus k_a are expressed as follows:

$$A_\theta(r, z) = \frac{\mu_0 I}{k_\mathrm{a}\pi}\sqrt{\frac{a}{r}}\left\{\left(1 - \frac{k_\mathrm{a}^2}{2}\right)K(k_\mathrm{a}) - E(k_\mathrm{a})\right\} \tag{44}$$

$$k_\mathrm{a}^2 = \frac{4ar}{(a + r)^2 + z^2} \tag{45}$$

where $K(k_\mathrm{a})$ and $E(k_\mathrm{a})$ are the complete elliptic integrals of first and second kinds with modulus k_a. In addition, the mutual inductance of two coaxial filamentary circles with radii r_1 and r_2, and distances between centres z_d, are expressed as follows:

$$M(r_1, r_2, z_\mathrm{d}) = \mu_0\sqrt{r_1 r_2}\left\{\left(\frac{2}{k_m} - k_m\right)K(k_m) - \frac{2}{k_m}E(k_m)\right\} \tag{46}$$

$$k_m^2 = \frac{4r_1 r_2}{(r_1 + r_2)^2 + z_\mathrm{d}^2}. \tag{47}$$

From equations (43) and (44), the self-inductance of the annular ring of a circular conductor can be numerically calculated.

For the self-inductance of the annular ring of a circular conductor with ring radius r and conductor radius a, the following expressions have been typically obtained by Kirchhoff, Rayleigh and Niven as follows [8, 9, 11, 12]:

$$L = \mu_0 r\left(\log\frac{8r}{a} - 1.75\right) \tag{48}$$

$$L = \mu_0 r\left\{\left(1 + \frac{a^2}{8r^2}\right)\log\frac{8r}{a} + \frac{a^2}{24r^2} - 1.75\right\}. \tag{49}$$

Equation (48) given by Kirchhoff can be obtained as the sum of the external and internal inductances by using equation (46) for the mutual inductance of the filamentary circular ring at the inner edge and the central circular ring as the external inductance [9, 11]. On the other hand, equation (49) has been more accurately obtained using a double surface integral equivalent to equation (40) for an expanded form of the mutual inductance of equation (46), given by Rayleigh and Niven [12].

In addition, the self-inductance can be calculated from the mean of two extreme mutual inductances, of the filamentary ring at the inner edge of radius $r - a$ and the central filamentary ring of radius r, and of the filamentary ring at the outer edge of radius $r + a$ and the central filamentary ring. Then, the self-inductance of the annular ring can be calculated as follows:

$$L_\mathrm{mean} = \frac{L_{(1)} + L_{(2)}}{2} = \frac{M(r, r-a, 0) + M(r, r+a, 0)}{2} + \frac{\mu_0 r}{4} \tag{50}$$

where $L_{(1)}$ and $L_{(2)}$ are the self-inductances calculated from the mutual inductance of the filamentary ring at the inner edge of radius $r - a$ or at the outer edge of radius $r + a$, and the central filamentary ring of radius r, respectively.

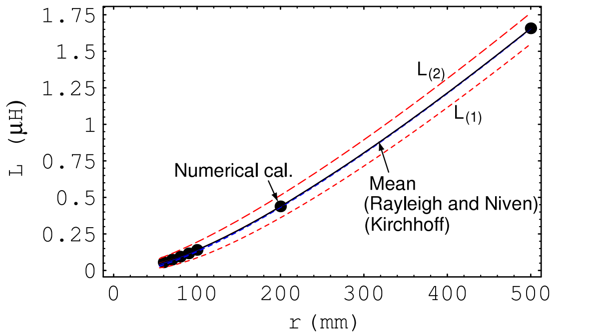

Figure 5. Self-inductance versus ring radius from 55 to 500 mm, for the annular ring of a circular conductor with radius $a = 50$ mm, together with the numerically and analytically calculated results.

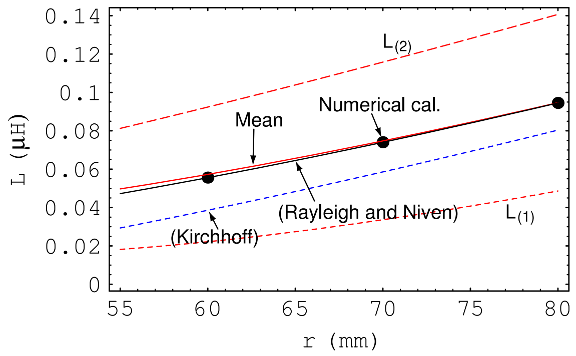

Figure 6. Self-inductance versus ring radius from 55 to 80 mm, for the annular ring of a circular conductor with conductor radius $a = 50$ mm.

A contour plot of $r \times A_\theta(r, z)$ and the vector plot of the magnetic field due to the annular ring of a circular conductor with ring radius $r = 100$ mm and conductor radius $a = 50$ mm are shown in figure 4. The ring radius dependence of the self-inductance for the annular ring of a circular conductor with conductor radius $a = 50$ mm is shown in figures 5 and 6. In figures 5 and 6, the numerical calculation using equation (43) and the analytical calculations using equation (48), the formula of Kirchhoff, and equation (49), the formula of Rayleigh and Niven, and using equation (50) for the simple mean are plotted, together with the calculated results for $L_{(1)}$ and $L_{(2)}$ which are plotted as dashed lines. From figure 5, it is seen that every analytical calculation has a good agreement with the numerical calculation for the large ring radius. From figure 6 for the small ring radius, it is shown that the approximation with the simple mean has a good agreement with the numerical calculation. However, equation (49), the formula of Rayleigh and Niven, is better than equation (50), the approximation of the simple mean.

In this paper, this calculation method with the simple mean is applied for the self-inductance of a helical conductor, as shown below.

4.2. Self-inductance of a helical conductor

If the external inductance L_e is approximated by the mean of two extreme mutual inductances of the inner helical filament located at $(r - a, \varphi, z = 0)$ and the central helical filament of the conductor passing through $(r, \varphi, z = 0)$, and the outer helical filament passing through $(r + a, \varphi, z = 0)$ and the central helical filament is taken, the self-inductance of a thin helical conductor can be calculated, using equation (24) as follows:

$$L = \frac{L_{(1)}+L_{(2)}}{2} = \frac{L_{e(1)}+L_{e(2)}}{2} + L_i = \frac{\mu_0 l}{2\pi}\left(\ln\frac{2l}{\sqrt{r(r+a)}} - 1\right) + \frac{\mu_0 l}{2\pi}\sum_{n=1}^{\infty}\{K_n(nkr)I_n(nk(r-a)) + K_n(nk(r+a)) \times I_n(nkr)\} + \frac{\mu_0 l}{8\pi}k^2\{(r-a)^2 + r^2\} + \frac{\mu_0 l}{8\pi}\sqrt{1+k^2r^2} + \frac{\mu_0 l}{4\pi}k^2 r(r-a)\sum_{n=1}^{\infty}\{K_{n+1}(nkr)I_{n+1}(nk(r-a)) + K_{n-1}(nkr)I_{n-1}(nk(r-a))\} + \frac{\mu_0 l}{4\pi}k^2 r(r+a) \times \sum_{n=1}^{\infty}\{K_{n+1}(nk(r+a))I_{n+1}(nkr) + K_{n-1}(nk(r+a)) \times I_{n-1}(nkr)\} \quad (51)$$

where the internal inductance L_i of a helical conductor can be calculated from the magnetic energy within the helical conductor, taking account of the length along the conductor, as follows:

$$L_i = \frac{\mu_0 l}{8\pi}\frac{\sqrt{l_p^2 + (2\pi r)^2}}{l_p} = \frac{\mu_0 l}{8\pi}\sqrt{1+k^2r^2} \quad (52)$$

where the cross-sectional shape perpendicular to the central axis of the helical conductor is always assumed as circular.

In the limit of $k \to 0$ (or $l_p \to \infty$), the above self-inductance becomes that of a long straight conductor as follows:

$$\lim_{k\to 0} L = \frac{\mu_0 l}{2\pi}\left(\ln\frac{2l}{\sqrt{r(r+a)}} - 1\right) + \frac{\mu_0 l}{2\pi}\sum_{n=1}^{\infty}\lim_{k\to 0}\{K_n(nkr) \times I_n(nk(r-a)) + K_n(nk(r+a))I_n(nkr)\} + \frac{\mu_0 l}{8\pi} = \frac{\mu_0 l}{2\pi}\left(\ln\frac{2l}{\sqrt{r(r+a)}} - 1\right) - \frac{\mu_0 l}{4\pi}\left(\ln\frac{a}{r} + \ln\frac{a}{r+a}\right) + \frac{\mu_0 l}{8\pi} = \frac{\mu_0 l}{2\pi}\left(\ln\frac{2l}{a} - \frac{3}{4}\right). \quad (53)$$

On the other hand, in another limit, $k \to \infty$ (or $l_p \to 0$), the self-inductance per unit length becomes the following result for $r \gg a$:

$$\lim_{k\to 0} L = \frac{\mu_0 l}{2\pi}\left(\ln\frac{2l}{\sqrt{r(r+a)}} - 1\right) + \frac{\mu_0 l}{8\pi}k^2\{(r-a)^2 + r^2\} \approx \frac{\mu_0 l}{2\pi}\left(\ln\frac{2l}{r} - 1\right) + \mu_0 n^2 \pi r^2 l. \quad (54)$$

As a result, it is shown that equation (54) is equal to equation (42) for $\Delta\varphi_1 = 2\pi$, i.e. the self-inductance of a long closely wound solenoid.

4.3. Self-inductance of a helical conductor or a solenoid as a closed loop

The inductance of a solenoid is discussed as that of a closed loop in most textbooks on classical electromagnetism. However, as mentioned above, it is reasonable to treat a twisted superconductor, e.g. a superconducting multifilamentary conductor, as a parallel circuit of unclosed helical conductors. Then, the self-inductance calculation for an unclosed helical conductor and a closed helical conductor (or solenoid with return conductor) must be distinguished.

Then, the self-inductance per unit length L_{closed}/l of an infinitely long solenoid of winding radius r and conductor radius a with the return conductor as a closed loop is calculated

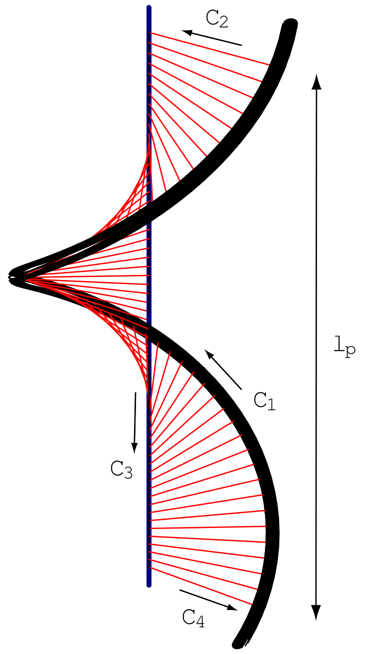

Figure 7. Whole integral path of C_1, C_2, C_3, and C_4 for the magnetic flux enclosed in one period of a closed solenoid, which is not identical to the substantial integral path C_1 for an unclosed solenoid or helical conductor.

for a closed path of C ($=C_1+C_2+C_3+C_4$), as shown in figure 7, as follows:

$$\frac{L_{\text{closed}}}{l} = \frac{L_e+L_i}{l} = \frac{\Phi_e}{l_p I} + \frac{L_i}{l} = \frac{1}{l_p I}\oint_c \vec{A}\cdot d\vec{s} + \frac{L_i}{l}$$
$$= \frac{1}{l_p I}\left(\int_{c_1}\vec{A}\cdot d\vec{s} + \int_{c_3}\vec{A}\cdot d\vec{s}\right) + \frac{L_i}{l}$$
$$= \frac{1}{l_p I}\left\{\int_0^{l_p} A_z(r-a,\theta,z)\,dz\right.$$
$$\left.+\int_0^{2\pi} A_\theta(r-a,\theta,z)(r-a)\,d\theta - \int_0^{l_p} A_z(0,\theta,z)\,dz\right\}$$
$$+\frac{L_i}{l} = \frac{\mu_0}{2\pi}k^2 r(r-a)\sum_{n=1}^{\infty}\{K_{n+1}(nkr)I_{n+1}(nk(r-a))$$
$$+K_{n-1}(nkr)I_{n-1}(nk(r-a))\}$$
$$+\frac{\mu_0}{\pi}\sum_{n=1}^{\infty}K_n(nkr)I_n(nk(r-a)) + \frac{\mu_0}{8\pi}\sqrt{1+k^2r^2}. \quad (55)$$

In the limit of $k\to\infty$ (or $l_p\to 0$), the self-inductance per unit length becomes the following result for $r \gg a$:

$$\lim_{k\to\infty}\frac{L_{\text{closed}}}{l} = \frac{\mu_0}{4\pi}k^2 r(r-a) \approx \mu_0 n^2 \pi r^2. \quad (56)$$

On the other hand, L_{closed}/l for another limit, $k\to 0$ (or $l_p\to\infty$), has no meaning, because the central axis loses the meaning in the limit of $k\to 0$ (or $l_p\to\infty$). Then, this means that the self-inductance of a straight conductor cannot be discussed by means of the calculation of the magnetic flux of the closed loop. It is thought that an unclosed helical conductor actually produces extra magnetic flux, which does not pass through the closed loop formed by $C=C_1+C_2+C_3+C_4$.

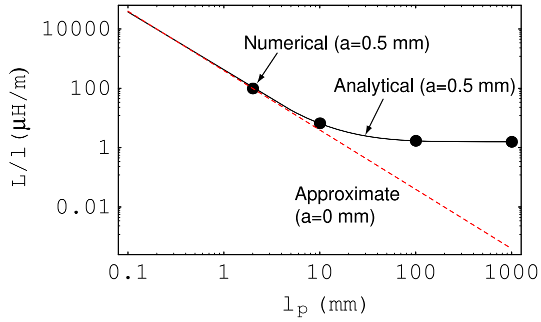

Figure 8. Pitch length dependence of self-inductance per unit length, L/l, versus l_p, for a long helical conductor of winding radius 10 mm, and conductor radius 0.5 mm.

The above calculated result obtained using equation (56) is well known as the self-inductance of a solenoid, but it is different from that of the unclosed helical conductor, as shown by equation (54). Furthermore, the self-inductance of a closed long solenoid consisting of a helical conductor with a return straight thin conductor at the z axis as shown in figure 7 can be also obtained, using equations (51), (53) and (24), with the equivalent result of equation (55).

5. Calculated results

For the circuit system consisting of a superconducting composite and a return conductor, the circuit equation for the transport current or the shielding current under the external field in a long superconductor is generally described without the length dependence due to the cancellation of the common term [1, 4, 13].

For the calculated results shown below, the intrinsic principal term $L_{12,r}$ is concentrated on, and it is presented without the subscript r.

5.1. A single helical conductor

The result calculated using equation (51) for the self-inductance of a helical conductor with the radius $a = 0.5$ mm and the winding radius $r = 10$ mm, is shown in figure 8, together with the conventional approximate self-inductance of the solenoid as a closed loop, $L/l \approx \mu_0 n^2\pi r^2 = \mu_0\pi r^2/l_p^2$, shown as a dashed line with the results shown as circular points, which were numerically calculated using equation (40). In this calculation, the analytical expressions up to the first hundred terms ($n = 100$) in equation (51) and up to the first fifty terms ($n = 50$) in the numerical integral calculation of equation (40) are taken into account. The same calculation of the self-inductance is plotted for the twist pitch parameter k in figure 9. It is shown that the agreement between the analytical and numerical calculations is quite good on both plots. The lower and upper dashed curves in figure 9 are calculated results for $L_{(1)}$ and $L_{(2)}$ of equation (51), respectively. Then, it is shown that the mean of $L_{(1)}$ and $L_{(2)}$ is much better than $L_{(1)}$ or $L_{(2)}$ as well as an annular ring.

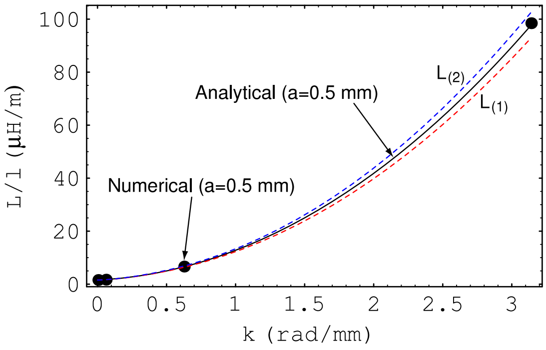

Figure 9. Pitch parameter dependence of self-inductance per unit length, L/l, versus k, for the same conductor as for figure 8.

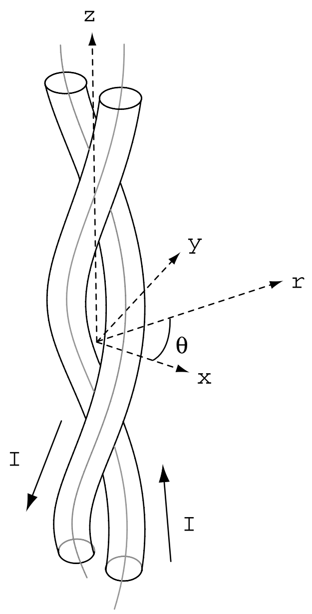

Figure 10. Schematic view of a twisted bifilar lead of a round conductor.

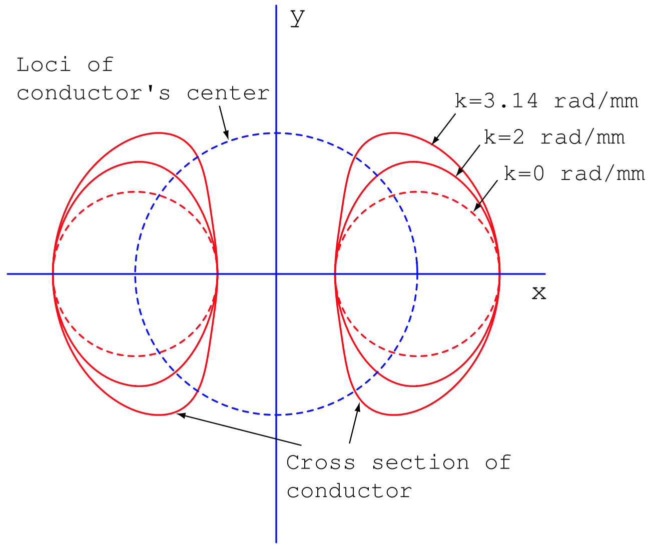

Figure 11. Cross-section at $z = 0$ perpendicular to the z axis for a helical bifilar lead of a round conductor, with $k = 0$, 2, and 3.14 rad mm^{-1}.

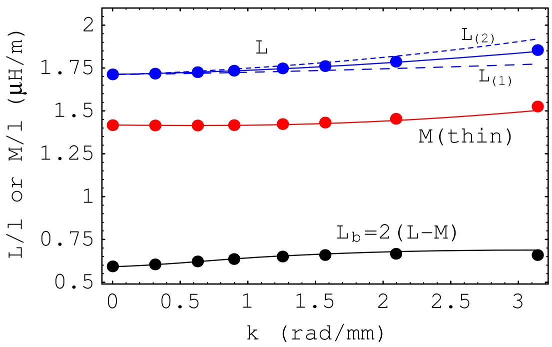

Figure 12. Twist dependence of the self- and mutual inductance L, M of each helical conductor, and the self-inductance $L_b = 2(L - M)$ of a twisted bifilar lead.

5.2. A helical bifilar lead

For a twisted bifilar lead, as shown in figures 10 and 11, with diameters of 0.49 mm, and distances of 0.84 mm, the self-inductance, L_b, of a twisted bifilar lead can be calculated as $L_b = 2(L - M)$ from the self-inductance, L, of each helical conductor and the mutual inductance, M, of the two helical conductors [1, 14]. The twist pitch dependences of the self-inductance, L_b, of a twisted bifilar lead and the self- and mutual inductances of helical conductors are shown in figure 12, with good agreement between the analytical and numerical calculations. The lower and upper dashed curves for the self-inductance are also calculated results, for $L_{(1)}$ and $L_{(2)}$, respectively. In addition, the mutual inductance is simply calculated using equation (24) instead of equation (40).

5.3. A twisted superconducting '6 around 1' strand cable

For a twisted superconducting 6 around 1 strand cable, the twist pitch dependence of the self- and mutual inductances of the strands of the twisted 6 around 1 strand cable have been calculated [1]. On the condition that the cross-sectional shape perpendicular to the helical central axis of the peripheral conductor is not circular except for $k = 0$, the agreement between the results, analytically calculated using equation (51) and numerically calculated using equation (40) was poor. The numerical calculation has been made taking account of the k dependence of the cross-sectional shape. Then, it is implied that equation (51) is not valid for a non-round conductor.

6. Conclusion

The full analytical expression for the mutual inductance of two coaxial long helical thin conductors for the whole range from 0 to ∞ of the pitch length was obtained from a modified form of Neumann's formula. It is shown that the expression obtained is equal to the intrinsic principal term of the mutual inductance previously obtained from the vector potential due to an infinitely long helical conductor, except for the common term. Then, the analytical expression for the self-inductance for a long helical round conductor was studied on the basis of the summation of the external and internal inductances. In addition, the external inductance of a helical conductor was calculated as the mean of the mutual inductances of the filament at the inner or outer edges of conductor and the central filament. It was confirmed that the results obtained by the approximate calculation show a good agreement with those obtained by the numerically calculation for a single helical round conductor.

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References

[1] Tominaka T 2005 Inductance calculation of helical conductors *Supercond. Sci. Technol.* **18** 214–22
[2] Tominaka T 2004 Vector potential for a single helical current conductor *Nucl. Instrum. Methods* A **523** 1–8
[3] Tominaka T 2006 Magnetic field calculation of an infinitely long solenoid *Eur. J. Phys.* **27** 1399–408
[4] Tominaka T 2005 Calculations using the helical filamentary structure for current distributions of twisted superconducting multifilamentary composites *Supercond. Sci. Technol.* **18** 634–43
[5] Gradshteyn I S and Ryzhik I M 2000 *Table of Integrals, Series, and Products* 6th edn (San Diego, CA: Academic) p 429 (eq 3.754(2))
[6] Watson G N 1944 *A Treatise on the Theory of Bessel Functions* 2nd edn (Cambridge: Cambridge University Press) p.172, (Sec. 6.16, eq(1)), p.185 (Sec. 6.3), p.361, (Sec. 8.3, eq(8))
[7] Abramowitz M and Stegun I A (ed) 1970 *Handbook of Mathematical Functions* (New York: Dover) pp 374–6
[8] Maxwell J C 1954 *A Treatise on Electricity and Magnetism* vol 2 (New York: Dover)
[9] Gray A 1921 *Absolute Measurements in Electricity and Magnetism'* 2nd edn (London: MacMillan)
[10] Sommerfeld A 1952 Electrodynamics *Lectures on Theoretical Physics* vol III (New York: Academic)
[11] Landau L D and Lifshitz E M 1960 *Electrodynamics of Continuous Media* (Reading, MA: Addison-Wesley)
[12] Lord Rayleigh 1912 On the self-induction of electric currents in a thin anchor-ring *Proc. R Soc.* A **86** 562–71
[13] Tominaka T 2006 Calculations using the circuit equation for current and field distributions of type-II superconductors *Supercond. Sci. Technol.* **19** 1040–6
[14] Tominaka T and Chiba Y 2004 Low frequency inductance for a twisted bifilar lead *J. Phys. D: Appl. Phys.* **37** 1592–5