This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2015.2402685, IEEE Transactions on Fuzzy Systems

Design of Polynomial Fuzzy Observer-Controller with Sampled-Output Measurements for Nonlinear Systems Considering Unmeasurable Premise Variables

Chuang Liu, and H.K. Lam, *Senior Member, IEEE*

***Abstract*—In this paper, we propose a polynomial fuzzy observer-controller for nonlinear systems where the design is achieved through the stability analysis of polynomial-fuzzy-model-based (PFMB) observer-control system. The polynomial fuzzy observer estimates the system states using estimated premise variables. The estimated states are then employed by the polynomial fuzzy controller for the feedback control of nonlinear systems represented by the polynomial fuzzy model. The system stability of the PFMB observer-control system is analyzed based on the Lyapunov stability theory. Although using estimated premise variables in polynomial fuzzy observer can handle a wider class of nonlinear systems, it leads to a significant drawback that the stability conditions obtained are non-convex. Matrix decoupling technique is employed to achieve convex stability conditions in the form of sum of squares (SOS). We further extend the design and analysis to polynomial fuzzy observer-controller using sampled-data technique for nonlinear systems where only sampled-output measurements are available. Simulation examples are presented to demonstrate the feasibility and validity of the design and analysis results.**

***Index Terms*—Polynomial fuzzy controller, polynomial fuzzy observer, sampled-output measurements, unmeasurable premise variables, sum of square (SOS).**

I. Introduction

TAKAGI-SUGENO (T-S) fuzzy model [1], [2] has been widely used as a modeling tool for nonlinear systems. It represents nonlinear systems as a combination of local linear subsystems weighted by membership functions. This particular modeling structure allows analysis techniques and control methods used for linear systems to be applied. Recently, polynomial fuzzy model [3], [4] was proposed to generalize the T-S fuzzy model. The modeling process is achieved by sector nonlinearity technique [5] which was extended using Taylor series expansion [6] to establish progressively precise polynomial fuzzy models. Based on the T-S or polynomial fuzzy model, Lyapunov stability theory [7] was employed as a mathematical tool to analyze the system stability. Stability conditions in terms of linear matrix inequalities (LMIs) [5], [8] and sum of squares (SOS) approach [9] are employed for T-S and polynomial fuzzy models, respectively, which can be numerically solved by convex programming techniques. Together with stability analysis, control synthesis can be achieved by the concept of parallel distributed compensation (PDC) [3], [7] and solving LMIs or SOS conditions rather than by predefining the feedback gains using trial-and-error or other design techniques (for example, pole placement).

Following the basic framework of fuzzy-model-based (FMB) stability analysis, three major research directions have been extensively investigated [10], [11]. The first direction is reducing the conservativeness of stability conditions by investigating the fuzzy summations. Due to the abandon of membership functions during the analysis, stability conditions are only sufficient but not necessary. To relax the stability conditions, the fuzzy summation was investigated in [12], [13] and further generalized by Pólya's theory in [14], [15]. The second direction is the variation of Lyapunov function candidates, for instance, quadratic Lyapunov function [7], switching Lyapunov function [16]–[18], fuzzy Lyapunov function [19]–[21], piecewise linear Lyapunov function [22], [23] and polynomial Lyapunov function [18], [20], [24]. The third direction is the membership-function-dependent analysis which brings the information of membership functions into stability analysis such as using symbolic variables [6], [25], [26], polynomial constraints [27], approximated membership functions [28], [29], and other techniques [21], [30]–[32]. Slack matrices are employed to carry the information of membership functions to stability conditions through S-procedure [33] at the expense of computational demand.

Based on the development of relaxed stability conditions, FMB control strategy is extended to control problems such as uncertainty [34], sampled-data system [35], [36] and output feedback [37]. Observer, being used in one of the output feedback control schemes, is exploited to estimate the states of systems when the output is only available for measuring. Fuzzy observer was proposed in [8] for the nonlinear system represented by the T-S fuzzy model. Under the restriction that the fuzzy model and fuzzy observer share the same set of premise membership functions depending on measurable premise system states, separation principle [38] can be applied to design the fuzzy controller and fuzzy observer independently. To widen the applicability of the fuzzy observer, the case that fuzzy observer with premise membership functions depending on estimated premise system states was considered in [39]. However, a two-step procedure was needed to solve the

The work described in this paper was partly supported by King's College London and China Scholarship Council.

C. Liu and H.K. Lam are with the Department of Informatics, King's College London, Strand, London, WC2R 2LC, United Kingdom. e-mail: {chuang.liu, hak-keung.lam}@kcl.ac.uk.

Manuscript received 2014.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2015.2402685, IEEE Transactions on Fuzzy Systems

non-convex stability conditions [39]. Various techniques such as matrix decoupling [40], completing squares [41], Finsler's lemma [42] and descriptor [43], were proposed to approximate the non-convex stability conditions by convex stability conditions such that convex programming techniques can be applied to numerically obtain feasible solutions. More recently, the fuzzy observer has been generalized to polynomial fuzzy observer [44]. However, only measurable premise variable was considered in the premise membership functions of both polynomial fuzzy model and polynomial fuzzy observer, and two steps were required to design the polynomial controller and observer gains. To the best of our knowledge, polynomial fuzzy observer with premise membership functions depending on unmeasurable premise variable has not been addressed.

Sampled-data control system is a control system whose states are measured only at the sampling instants. The zero-order-hold unit keeps the control signal constant between sampling instants, which complicates the system dynamics and makes the stability analysis much more difficult. Various methods were proposed to investigate the stability of sampled-data control system such as lifting technique [45], hybrid discrete/continuous approach [46], input-delay approach [47] and exact discrete-time design approach [48]. Among these approaches, input-delay approach represents the discrete-time input measurements into time-delayed input measurements, and makes continuous-time stability analysis applicable to sampled-data control systems. Combined with FMB control, fruitful results were obtained [49]–[53] for full-state feedback case. Sampled-data fuzzy observer-controller receives much less attention because of its complexities on stability analysis. Fuzzy observer [54]–[56] or dynamic output feedback [57], [58] using sampled-output measurements can be found in the literature for nonlinear systems represented by T-S fuzzy models. To the best of our knowledge, polynomial fuzzy observer has not been applied to systems with sampled-output measurements.

Although polynomial fuzzy observer-controller and sampled-data polynomial fuzzy observer-controller are relatively less investigated, they are vital to the nonlinear control systems when full states are not available for performing feedback control. It motivates us to investigate the system stability of polynomial-fuzzy-model-based (PFMB) observer-control systems. We consider the polynomial fuzzy controller and polynomial fuzzy observer whose premise membership functions depend on estimated premise variables. Matrix decoupling technique [40] is employed to achieve convex SOS-based stability conditions. Moreover, we consider the polynomial fuzzy observer using sampled-output measurements for state estimation. Input-delay approach [47] is employed to investigate the system stability.

This paper is organized as follows. In Section II, notations and the formulation of polynomial fuzzy model, polynomial fuzzy observer and polynomial fuzzy controller are described. In Section III, stability analysis is conducted for PFMB observer-control system and further for systems under sampled-output measurements. In Section IV, simulation examples are provided to demonstrate the feasibility and validity of stability conditions. In Section V, a conclusion is drawn.

II. Preliminary

A. Notation

The following notation is employed throughout this paper [9]. A monomial in $\mathbf{x}(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T$ is a function of the form $x_1^{d_1}(t)x_2^{d_2}(t)\cdots x_n^{d_n}(t)$, where $d_i \geq 0, i = 1, 2, \ldots, n$, are integers. The degree of a monomial is $d = \sum_{i=1}^{n} d_i$. A polynomial $\mathbf{p}(\mathbf{x}(t))$ is a finite linear combination of monomials with real coefficients. A polynomial $\mathbf{p}(\mathbf{x}(t))$ is an SOS if it can be written as $\mathbf{p}(\mathbf{x}(t)) = \sum_{j=1}^{m} \mathbf{q}_j(\mathbf{x}(t))^2$, where $\mathbf{q}_j(\mathbf{x}(t))$ is a polynomial and m is a nonnegative integer. It can be concluded that if $\mathbf{p}(\mathbf{x}(t))$ is an SOS, then $\mathbf{p}(\mathbf{x}(t)) \geq 0$. The expressions of $\mathbf{M} > 0, \mathbf{M} \geq 0, \mathbf{M} < 0$, and $\mathbf{M} \leq 0$ denote the positive, semi-positive, negative, and semi-negative definite matrices $\mathbf{M}$, respectively. The symbol "*" in a matrix represents the transposed element in the corresponding position.

B. Polynomial Fuzzy Model

The i^{th} rule of the polynomial fuzzy model for the nonlinear system is presented as follows [3]:

$$\begin{aligned}\text{Rule } i: &\text{ IF } f_1(\mathbf{x}(t)) \text{ is } M_1^i \text{ AND} \cdots \text{AND } f_\Psi(\mathbf{x}(t)) \text{ is } M_\Psi^i,\\ &\text{THEN } \dot{\mathbf{x}}(t) = \mathbf{A}_i(\mathbf{x}(t))\mathbf{x}(t) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{u}(t),\\ &\qquad \mathbf{y}(t) = \mathbf{C}_i(\mathbf{x}(t))\mathbf{x}(t),\end{aligned}$$

where $\mathbf{x}(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T$ is the state vector, and n is the dimension of the nonlinear system; $f_\eta(\mathbf{x}(t))$ is the premise variable corresponding to its fuzzy term M_η^i in rule i, $\eta = 1, 2, \ldots, \Psi$, and Ψ is a positive integer; $\mathbf{A}_i(\mathbf{x}(t)) \in \Re^{n\times N}$ and $\mathbf{B}_i(\mathbf{x}(t)) \in \Re^{n\times m}$ are the known polynomial system and input matrices, respectively; $\mathbf{u}(t) \in \Re^m$ is the control input vector; $\mathbf{y}(t) \in \Re^l$ is the output vector; $\mathbf{C}_i(\mathbf{x}(t)) \in \Re^{l\times N}$ is the polynomial output matrix. The dynamics of the nonlinear system is given by

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \sum_{i=1}^{p} w_i(\mathbf{x}(t))\Big(\mathbf{A}_i(\mathbf{x}(t))\mathbf{x}(t) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{u}(t)\Big),\\ \mathbf{y}(t) &= \sum_{i=1}^{p} w_i(\mathbf{x}(t))\mathbf{C}_i(\mathbf{x}(t))\mathbf{x}(t),\end{aligned} \tag{1}$$

where p is the number of rules in the polynomial fuzzy model; $w_i(\mathbf{x}(t))$ is the normalized grade of membership, $w_i(\mathbf{x}(t)) = \dfrac{\prod_{\eta=1}^{\Psi} \mu_{M_\eta^i}(f_\eta(\mathbf{x}(t)))}{\sum_{k=1}^{p}\prod_{\eta=1}^{\Psi} \mu_{M_\eta^k}(f_\eta(\mathbf{x}(t)))}$, $w_i(\mathbf{x}(t)) \geq 0, i = 1, 2, \ldots, p$, and $\sum_{i=1}^{p} w_i(\mathbf{x}(t)) = 1$; $\mu_{M_\eta^i}(f_\eta(\mathbf{x}(t))), \eta = 1, 2, \ldots, \Psi$, are grades of membership corresponding to the fuzzy term M_η^i.

C. Polynomial Fuzzy Observer

For brevity, time t is dropped from now. Considering premise variable $f_\eta(\mathbf{x})$ depending on unmeasurable states $\mathbf{x}$, we apply the following polynomial fuzzy observer to estimate the states in (1). The i^{th} rule of the polynomial fuzzy observer is described as follows:

$$\begin{aligned}\text{Rule } i: &\text{ IF } f_1(\breve{\mathbf{x}}) \text{ is } M_1^i \text{ AND} \cdots \text{AND } f_\Psi(\breve{\mathbf{x}}) \text{ is } M_\Psi^i,\\ &\text{THEN } \dot{\breve{\mathbf{x}}} = \mathbf{A}_i(\breve{\mathbf{x}})\breve{\mathbf{x}} + \mathbf{B}_i(\breve{\mathbf{x}})\mathbf{u} + \mathbf{L}_i(\breve{\mathbf{x}})(\mathbf{y} - \breve{\mathbf{y}}),\\ &\qquad \breve{\mathbf{y}} = \mathbf{C}_i(\breve{\mathbf{x}})\breve{\mathbf{x}},\end{aligned}$$

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2015.2402685, IEEE Transactions on Fuzzy Systems

where $\check{\mathbf{x}} \in \Re^n$ is the estimated state $\mathbf{x}$; $\check{\mathbf{y}} \in \Re^l$ is the estimated output $\mathbf{y}$; $\mathbf{L}_i(\check{\mathbf{x}}) \in \Re^{N\times l}$ is the polynomial observer gain. The polynomial fuzzy observer is given by

$$\begin{aligned}\dot{\check{\mathbf{x}}} &= \sum_{i=1}^{p} w_i(\check{\mathbf{x}})\Big(\mathbf{A}_i(\check{\mathbf{x}})\check{\mathbf{x}} + \mathbf{B}_i(\check{\mathbf{x}})\mathbf{u} + \mathbf{L}_i(\check{\mathbf{x}})(\mathbf{y} - \check{\mathbf{y}})\Big),\\ \check{\mathbf{y}} &= \sum_{i=1}^{p} w_i(\check{\mathbf{x}})\mathbf{C}_i(\check{\mathbf{x}})\check{\mathbf{x}}. \qquad (2)\end{aligned}$$

It can be seen from (2) that the membership functions of polynomial fuzzy observer depend on estimated system states $\check{\mathbf{x}}$ rather than original system states $\mathbf{x}$.

D. Polynomial Fuzzy Controller

With PDC design approach [3], [7], the i^{th} rule of the polynomial fuzzy controller is described as follows:

$$\begin{aligned}&\text{Rule } i: \text{IF } f_1(\check{\mathbf{x}}) \text{ is } M_1^i \text{ AND} \cdots \text{AND } f_\Psi(\check{\mathbf{x}}) \text{ is } M_\Psi^i,\\ &\qquad \text{THEN } \mathbf{u} = \mathbf{G}_i(\check{\mathbf{x}})\check{\mathbf{x}},\end{aligned}$$

where $\mathbf{G}_i(\check{\mathbf{x}}) \in \Re^{m\times N}$ is the polynomial controller gain. The polynomial fuzzy controller is given by

$$\mathbf{u} = \sum_{i=1}^{p} w_i(\check{\mathbf{x}})\mathbf{G}_i(\check{\mathbf{x}})\check{\mathbf{x}}. \qquad (3)$$

Note that in (3) both the premise variable and the controller gain depend on estimated states $\check{\mathbf{x}}$.

E. Useful Lemmas

The following lemmas are employed in this paper.

Lemma 1: With $\mathbf{X}, \mathbf{Y}$ of appropriate dimension and $\beta > 0$, the following inequality holds [59]:

$$\mathbf{X}^T\mathbf{Y} + \mathbf{Y}^T\mathbf{X} \le \beta\mathbf{X}^T\mathbf{X} + \frac{1}{\beta}\mathbf{Y}^T\mathbf{Y}.$$

Lemma 2: With $\mathbf{P}, \mathbf{Q}$ of appropriate dimension, $\mathbf{Q} > 0$ and a scalar γ, the following inequality holds [59]:

$$-\mathbf{P}\mathbf{Q}^{-1}\mathbf{P} \le \gamma^2\mathbf{Q} - 2\gamma(\mathbf{P}^T + \mathbf{P}).$$

Lemma 3 (Jensen's inequality): With $\mathbf{x}(t), \mathbf{Q}$ of appropriate dimension, $\mathbf{Q} > 0$ and $h > 0$, the following inequality holds [60]:

$$\begin{aligned}&-h\int_{t-h}^{t} \dot{\mathbf{x}}(\varphi)^T\mathbf{Q}\dot{\mathbf{x}}(\varphi)d\varphi\\ &\le -(\mathbf{x}(t) - \mathbf{x}(t-h))^T\mathbf{Q}(\mathbf{x}(t) - \mathbf{x}(t-h)).\end{aligned}$$

III. Stability Analysis

In this section, the stability analysis is carried out for PFMB observer-control systems. The formulation of closed-loop PFMB observer-control systems are provided first. Then based on Lyapunov stability theory, stability conditions are obtained in terms of SOS. Matrix decoupling technique is employed to obtain convex SOS-based stability conditions. Finally, using similar techniques, we extend the stability analysis to systems with sampled-output measurement.

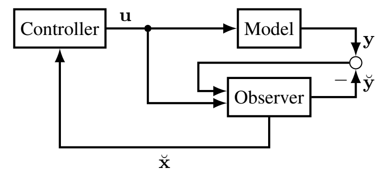


Fig. 1. A block diagram of PFMB observer-control systems.

A. Polynomial Fuzzy Controller and Observer

The estimation error is defined as $\mathbf{e} = \mathbf{x} - \check{\mathbf{x}}$, and then we have the closed-loop system (shown in Fig. 1) consisting of the polynomial fuzzy model (1), the polynomial fuzzy controller (3) and the polynomial fuzzy observer (2) as follows:

$$\begin{aligned}\dot{\mathbf{x}} &= \sum_{i=1}^{p}\sum_{j=1}^{p} w_i(\mathbf{x})w_j(\check{\mathbf{x}})\Big((\mathbf{A}_i(\mathbf{x}) + \mathbf{B}_i(\mathbf{x})\mathbf{G}_j(\check{\mathbf{x}}))\check{\mathbf{x}}\\ &\quad + \mathbf{A}_i(\mathbf{x})\mathbf{e}\Big), \qquad (4)\\ \dot{\check{\mathbf{x}}} &= \sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p} w_i(\mathbf{x})w_j(\check{\mathbf{x}})w_k(\check{\mathbf{x}})\Big((\mathbf{A}_j(\check{\mathbf{x}}) + \mathbf{B}_j(\check{\mathbf{x}})\mathbf{G}_k(\check{\mathbf{x}})\\ &\quad + \mathbf{L}_j(\check{\mathbf{x}})(\mathbf{C}_i(\mathbf{x}) - \mathbf{C}_k(\check{\mathbf{x}})))\check{\mathbf{x}} + \mathbf{L}_j(\check{\mathbf{x}})\mathbf{C}_i(\mathbf{x})\mathbf{e}\Big), \qquad (5)\\ \dot{\mathbf{e}} &= \sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p} w_i(\mathbf{x})w_j(\check{\mathbf{x}})w_k(\check{\mathbf{x}})\Big((\mathbf{A}_i(\mathbf{x}) - \mathbf{A}_j(\check{\mathbf{x}})\\ &\quad + (\mathbf{B}_i(\mathbf{x}) - \mathbf{B}_j(\check{\mathbf{x}}))\mathbf{G}_k(\check{\mathbf{x}}) - \mathbf{L}_j(\check{\mathbf{x}})(\mathbf{C}_i(\mathbf{x}) - \mathbf{C}_k(\check{\mathbf{x}})))\check{\mathbf{x}}\\ &\quad + (\mathbf{A}_i(\mathbf{x}) - \mathbf{L}_j(\check{\mathbf{x}})\mathbf{C}_i(\mathbf{x}))\mathbf{e}\Big). \qquad (6)\end{aligned}$$

The control objective is to make the augmented observer-control system ((5) and (6)) asymptotically stable, i.e., $\check{\mathbf{x}} \to 0$ and $\mathbf{e} \to 0$ as time $t \to \infty$, by determining the polynomial controller gain $\mathbf{G}_k(\check{\mathbf{x}})$ and polynomial observer gain $\mathbf{L}_j(\check{\mathbf{x}})$.

Theorem 1: The augmented PFMB observer-control system (formed by (5) and (6)) is guaranteed to be asymptotically stable if there exist matrices $\mathbf{X} \in \Re^{N\times N}, \mathbf{Y} \in \Re^{N\times N}, \mathbf{N}_k(\check{\mathbf{x}}) \in \Re^{m\times N}, \mathbf{M}_j(\check{\mathbf{x}}) \in \Re^{N\times l}, k = 1, 2, \ldots, p, j = 1, 2, \ldots, p$, and predefined scalers $\alpha_1 > 0, \alpha_2 > 0, \beta > 0$ such that the following SOS-based conditions are satisfied:

$$\begin{aligned}&\nu^T(\mathbf{X} - \varepsilon_1\mathbf{I})\nu \text{ is SOS}; \qquad (7)\\ &\nu^T(\mathbf{Y} - \varepsilon_2\mathbf{I})\nu \text{ is SOS}; \qquad (8)\\ &-\nu^T(\mathbf{\Phi}_{ijk}(\mathbf{x},\check{\mathbf{x}}) + \mathbf{\Phi}_{ikj}(\mathbf{x},\check{\mathbf{x}}) + \varepsilon_3(\mathbf{x},\check{\mathbf{x}})\mathbf{I})\nu \text{ is SOS}\\ &\forall i, j \le k; \qquad (9)\\ &-\nu^T(\mathbf{\Theta}_{ijk}(\mathbf{x},\check{\mathbf{x}}) + \mathbf{\Theta}_{ikj}(\mathbf{x},\check{\mathbf{x}}) + \varepsilon_4(\mathbf{x},\check{\mathbf{x}})\mathbf{I})\nu \text{ is SOS}\\ &\forall i, j \le k; \qquad (10)\end{aligned}$$

where

$$\mathbf{\Phi}_{ijk}(\mathbf{x},\check{\mathbf{x}}) = \begin{bmatrix} \tilde{\mathbf{\Gamma}}_{ijk}(\mathbf{x},\check{\mathbf{x}}) & \mathbf{\Phi}^{(12)} & \mathbf{\Phi}^{(13)}\\ * & -\frac{1}{\alpha_1}\mathbf{Y} & \mathbf{0}\\ * & * & -\frac{1}{\beta}\mathbf{I}\end{bmatrix}, \qquad (11)$$

$$\mathbf{\Theta}_{ijk}(\mathbf{x},\check{\mathbf{x}}) = \begin{bmatrix} \tilde{\mathbf{\Lambda}}_{ij}(\mathbf{x},\check{\mathbf{x}}) & \mathbf{\Theta}^{(12)} & \tilde{\mathbf{\Theta}}_{ijk}^{(13)}(\mathbf{x},\check{\mathbf{x}})\\ * & -\frac{1}{\alpha_2}\mathbf{I} & \mathbf{0}\\ * & * & -\beta\mathbf{I}\end{bmatrix}, \qquad (12)$$

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2015.2402685, IEEE Transactions on Fuzzy Systems

$$\tilde{\mathbf{\Gamma}}_{ijk}(\mathbf{x},\breve{\mathbf{x}}) = \begin{bmatrix} \hat{\mathbf{\Xi}}_{jk}^{(11)}(\breve{\mathbf{x}}) + \hat{\mathbf{\Xi}}_{jk}^{(11)}(\breve{\mathbf{x}})^T & \hat{\mathbf{\Xi}}_{ijk}^{(21)}(\mathbf{x},\breve{\mathbf{x}})^T \\ * & -\alpha_2\mathbf{I} \end{bmatrix}, \tag{13}$$

$$\tilde{\mathbf{\Lambda}}_{ij}(\mathbf{x},\breve{\mathbf{x}}) = \begin{bmatrix} -\alpha_1\mathbf{Y} & \tilde{\mathbf{\Xi}}_{ij}^{(12)}(\mathbf{x},\breve{\mathbf{x}}) \\ * & \tilde{\mathbf{\Xi}}_{ij}^{(22)}(\mathbf{x},\breve{\mathbf{x}}) + \tilde{\mathbf{\Xi}}_{ij}^{(22)}(\mathbf{x},\breve{\mathbf{x}})^T \end{bmatrix}, \tag{14}$$

$$\mathbf{\Phi}^{(12)} = [\mathbf{I} \quad \mathbf{0}]^T, \tag{15}$$

$$\mathbf{\Phi}^{(13)} = [\mathbf{X} \quad \mathbf{0}]^T, \tag{16}$$

$$\mathbf{\Theta}^{(12)} = [\mathbf{0} \quad \mathbf{Y}]^T, \tag{17}$$

$$\tilde{\mathbf{\Theta}}_{ijk}^{(13)}(\mathbf{x},\breve{\mathbf{x}}) = [\tilde{\mathbf{H}}_{ijk}(\mathbf{x},\breve{\mathbf{x}})^T \quad -\tilde{\mathbf{H}}_{ijk}(\mathbf{x},\breve{\mathbf{x}})^T]^T, \tag{18}$$

$$\hat{\mathbf{\Xi}}_{jk}^{(11)}(\breve{\mathbf{x}}) = \mathbf{A}_j(\breve{\mathbf{x}})\mathbf{X} + \mathbf{B}_j(\breve{\mathbf{x}})\mathbf{N}_k(\breve{\mathbf{x}}), \tag{19}$$

$$\hat{\mathbf{\Xi}}_{ijk}^{(21)}(\mathbf{x},\breve{\mathbf{x}}) = (\mathbf{A}_i(\mathbf{x}) - \mathbf{A}_j(\breve{\mathbf{x}}))\mathbf{X} + (\mathbf{B}_i(\mathbf{x}) - \mathbf{B}_j(\breve{\mathbf{x}}))\mathbf{N}_k(\breve{\mathbf{x}}), \tag{20}$$

$$\tilde{\mathbf{\Xi}}_{ij}^{(12)}(\mathbf{x},\breve{\mathbf{x}}) = \mathbf{M}_j(\breve{\mathbf{x}})\mathbf{C}_i(\mathbf{x}), \tag{21}$$

$$\tilde{\mathbf{\Xi}}_{ij}^{(22)}(\mathbf{x},\breve{\mathbf{x}}) = \mathbf{Y}\mathbf{A}_i(\mathbf{x}) - \mathbf{M}_j(\breve{\mathbf{x}})\mathbf{C}_i(\mathbf{x}), \tag{22}$$

$$\tilde{\mathbf{H}}_{ijk}(\mathbf{x},\breve{\mathbf{x}}) = \mathbf{M}_j(\breve{\mathbf{x}})(\mathbf{C}_i(\mathbf{x}) - \mathbf{C}_k(\breve{\mathbf{x}})); \tag{23}$$

ν is an arbitrary vector independent of $\mathbf{x}$ with appropriate dimensions; $\varepsilon_1 > 0, \varepsilon_2 > 0, \varepsilon_3(\mathbf{x},\breve{\mathbf{x}}) > 0$ and $\varepsilon_4(\mathbf{x},\breve{\mathbf{x}}) > 0$ are predefined scalar polynomials; and the polynomial controller and observer gains are given by $\mathbf{G}_k(\breve{\mathbf{x}}) = \mathbf{N}_k(\breve{\mathbf{x}})\mathbf{X}^{-1}$ and $\mathbf{L}_j(\breve{\mathbf{x}}) = \mathbf{Y}^{-1}\mathbf{M}_j(\breve{\mathbf{x}})$, respectively.

Proof: Defining the augmented vector $\mathbf{z} = [\breve{\mathbf{x}}^T \quad \mathbf{e}^T]^T$ and the summation term $\sum_{i,j,k=1}^{p}\tilde{w}_{ijk} = \sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p} w_i(\mathbf{x})w_j(\breve{\mathbf{x}})w_k(\breve{\mathbf{x}})$, the augmented PFMB observer-control system is written as

$$\dot{\mathbf{z}} = \sum_{i,j,k=1}^{p}\tilde{w}_{ijk}\mathbf{\Xi}_{ijk}(\mathbf{x},\breve{\mathbf{x}})\mathbf{z}, \tag{24}$$

where

$$\mathbf{\Xi}_{ijk}(\mathbf{x},\breve{\mathbf{x}}) = \begin{bmatrix} \mathbf{\Xi}_{jk}^{(11)}(\breve{\mathbf{x}}) + \mathbf{H}_{ijk}(\mathbf{x},\breve{\mathbf{x}}) & \mathbf{\Xi}_{ij}^{(12)}(\mathbf{x},\breve{\mathbf{x}}) \\ \mathbf{\Xi}_{ijk}^{(21)}(\mathbf{x},\breve{\mathbf{x}}) - \mathbf{H}_{ijk}(\mathbf{x},\breve{\mathbf{x}}) & \mathbf{\Xi}_{ij}^{(22)}(\mathbf{x},\breve{\mathbf{x}}) \end{bmatrix}, \tag{25}$$

$$\mathbf{\Xi}_{jk}^{(11)}(\breve{\mathbf{x}}) = \mathbf{A}_j(\breve{\mathbf{x}}) + \mathbf{B}_j(\breve{\mathbf{x}})\mathbf{G}_k(\breve{\mathbf{x}}), \tag{26}$$

$$\mathbf{\Xi}_{ijk}^{(21)}(\mathbf{x},\breve{\mathbf{x}}) = \mathbf{A}_i(\mathbf{x}) - \mathbf{A}_j(\breve{\mathbf{x}}) + (\mathbf{B}_i(\mathbf{x}) - \mathbf{B}_j(\breve{\mathbf{x}}))\mathbf{G}_k(\breve{\mathbf{x}}), \tag{27}$$

$$\mathbf{\Xi}_{ij}^{(12)}(\mathbf{x},\breve{\mathbf{x}}) = \mathbf{L}_j(\breve{\mathbf{x}})\mathbf{C}_i(\mathbf{x}), \tag{28}$$

$$\mathbf{\Xi}_{ij}^{(22)}(\mathbf{x},\breve{\mathbf{x}}) = \mathbf{A}_i(\mathbf{x}) - \mathbf{L}_j(\breve{\mathbf{x}})\mathbf{C}_i(\mathbf{x}), \tag{29}$$

$$\mathbf{H}_{ijk}(\mathbf{x},\breve{\mathbf{x}}) = \mathbf{L}_j(\breve{\mathbf{x}})(\mathbf{C}_i(\mathbf{x}) - \mathbf{C}_k(\breve{\mathbf{x}})). \tag{30}$$

The following Lyapunov function candidate is employed to investigate the stability of the augmented PFMB observer-control system (24):

$$V(\mathbf{z}) = \mathbf{z}^T\mathbf{P}\mathbf{z}, \tag{31}$$

where $\mathbf{P} = \begin{bmatrix} \mathbf{X}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{Y} \end{bmatrix}, \mathbf{X} > 0, \mathbf{Y} > 0$, and thus $\mathbf{P} > 0$.

The time derivative of Lyapunov function is

$$\dot{V}(\mathbf{z}) = \sum_{i,j,k=1}^{p}\tilde{w}_{ijk}\mathbf{z}^T(\mathbf{P}\mathbf{\Xi}_{ijk}(\mathbf{x},\breve{\mathbf{x}}) + \mathbf{\Xi}_{ijk}(\mathbf{x},\breve{\mathbf{x}})^T\mathbf{P})\mathbf{z}. \tag{32}$$

Therefore, $\dot{V}(\mathbf{z}) < 0$ holds if

$$\sum_{i,j,k=1}^{p}\tilde{w}_{ijk}(\mathbf{P}\mathbf{\Xi}_{ijk}(\mathbf{x},\breve{\mathbf{x}}) + \mathbf{\Xi}_{ijk}(\mathbf{x},\breve{\mathbf{x}})^T\mathbf{P}) < 0. \tag{33}$$

Remark 1: The augmented PFMB observer-control system (24) is guaranteed to be asymptotically stable if $V(\mathbf{z}) > 0$ by satisfying $\mathbf{P} > 0$ and $\dot{V}(\mathbf{z}) < 0$ by satisfying (33) excluding $\mathbf{x} = \mathbf{0}$. It should be noted that the condition (33) is not convex. If the condition (33) is applied, the polynomial fuzzy controller gain $\mathbf{G}_k(\breve{\mathbf{x}})$ and polynomial fuzzy observer gain $\mathbf{L}_j(\breve{\mathbf{x}})$ are needed to be pre-determined.

In the following, we apply congruence transformation and matrix decoupling technique to obtain convex SOS stability conditions such that the polynomial fuzzy controller gain $\mathbf{G}_k(\breve{\mathbf{x}})$ and polynomial fuzzy observer gain $\mathbf{L}_j(\breve{\mathbf{x}})$ can be obtained using convex programming techniques.

Performing congruence transformation to (33) by pre-multiplying and post-multiplying $\mathbf{P}^{-1} = \begin{bmatrix} \mathbf{X} & \mathbf{0} \\ \mathbf{0} & \mathbf{Y}^{-1} \end{bmatrix}$ to both sides and denoting $\mathbf{N}_k(\breve{\mathbf{x}}) = \mathbf{G}_k(\breve{\mathbf{x}})\mathbf{X}$, we have

$$\sum_{i,j,k=1}^{p}\tilde{w}_{ijk}(\hat{\mathbf{\Xi}}_{ijk}(\mathbf{x},\breve{\mathbf{x}}) + \hat{\mathbf{\Xi}}_{ijk}(\mathbf{x},\breve{\mathbf{x}})^T) < 0, \tag{34}$$

where

$$\hat{\mathbf{\Xi}}_{ijk}(\mathbf{x},\breve{\mathbf{x}}) = \begin{bmatrix} \hat{\mathbf{\Xi}}_{jk}^{(11)}(\breve{\mathbf{x}}) + \mathbf{H}_{ijk}(\mathbf{x},\breve{\mathbf{x}})\mathbf{X} & \hat{\mathbf{\Xi}}_{ij}^{(12)}(\mathbf{x},\breve{\mathbf{x}}) \\ \hat{\mathbf{\Xi}}_{ijk}^{(21)}(\mathbf{x},\breve{\mathbf{x}}) - \mathbf{H}_{ijk}(\mathbf{x},\breve{\mathbf{x}})\mathbf{X} & \hat{\mathbf{\Xi}}_{ij}^{(22)}(\mathbf{x},\breve{\mathbf{x}}) \end{bmatrix}, \tag{35}$$

$$\hat{\mathbf{\Xi}}_{ij}^{(12)}(\mathbf{x},\breve{\mathbf{x}}) = \mathbf{L}_j(\breve{\mathbf{x}})\mathbf{C}_i(\mathbf{x})\mathbf{Y}^{-1}, \tag{36}$$

$$\hat{\mathbf{\Xi}}_{ij}^{(22)}(\mathbf{x},\breve{\mathbf{x}}) = \mathbf{A}_i(\mathbf{x})\mathbf{Y}^{-1} - \mathbf{L}_j(\breve{\mathbf{x}})\mathbf{C}_i(\mathbf{x})\mathbf{Y}^{-1}, \tag{37}$$

$\hat{\mathbf{\Xi}}_{jk}^{(11)}(\breve{\mathbf{x}})$ and $\hat{\mathbf{\Xi}}_{ijk}^{(21)}(\mathbf{x},\breve{\mathbf{x}})$ are defined in (19) and (20), respectively.

Applying Lemma 1, we have

$$\begin{aligned} &\sum_{i,j,k=1}^{p}\tilde{w}_{ijk}(\hat{\mathbf{\Xi}}_{ijk}(\mathbf{x},\breve{\mathbf{x}}) + \hat{\mathbf{\Xi}}_{ijk}(\mathbf{x},\breve{\mathbf{x}})^T) \\ \le &\sum_{i,j,k=1}^{p}\tilde{w}_{ijk}\Big(\mathbf{\Upsilon}_{ijk}(\mathbf{x},\breve{\mathbf{x}}) + \beta\mathbf{\Phi}^{(13)}(\mathbf{\Phi}^{(13)})^T\Big) \\ &+ \frac{1}{\beta}\Big(\sum_{i,j,k=1}^{p}\tilde{w}_{ijk}\mathbf{\Theta}_{ijk}^{(13)}(\mathbf{x},\breve{\mathbf{x}})\Big)\Big(\sum_{i,j,k=1}^{p}\tilde{w}_{ijk}\mathbf{\Theta}_{ijk}^{(13)}(\mathbf{x},\breve{\mathbf{x}})\Big)^T, \end{aligned} \tag{38}$$

where

$$\mathbf{\Upsilon}_{ijk}(\mathbf{x},\breve{\mathbf{x}}) = \begin{bmatrix} \hat{\mathbf{\Xi}}_{jk}^{(11)}(\breve{\mathbf{x}}) + \hat{\mathbf{\Xi}}_{jk}^{(11)}(\breve{\mathbf{x}})^T & \hat{\mathbf{\Xi}}_{ij}^{(12)}(\mathbf{x},\breve{\mathbf{x}}) + \hat{\mathbf{\Xi}}_{ijk}^{(21)}(\mathbf{x},\breve{\mathbf{x}})^T \\ * & \hat{\mathbf{\Xi}}_{ij}^{(22)}(\mathbf{x},\breve{\mathbf{x}}) + \hat{\mathbf{\Xi}}_{ij}^{(22)}(\mathbf{x},\breve{\mathbf{x}})^T \end{bmatrix}, \tag{39}$$

$$\mathbf{\Theta}_{ijk}^{(13)}(\mathbf{x},\breve{\mathbf{x}}) = [\mathbf{H}_{ijk}(\mathbf{x},\breve{\mathbf{x}})^T \quad -\mathbf{H}_{ijk}(\mathbf{x},\breve{\mathbf{x}})^T]^T, \tag{40}$$

$\mathbf{\Phi}^{(13)}$ is defined in (16).

Using matrix decoupling technique [40] to further separate decision variables in order to obtain convex SOS stability conditions, we rewrite $\mathbf{\Upsilon}_{ijk}(\mathbf{x}, \breve{\mathbf{x}})$ as follows:

$$\mathbf{\Upsilon}_{ijk}(\mathbf{x}, \breve{\mathbf{x}}) = \mathbf{\Gamma}_{ijk}(\mathbf{x}, \breve{\mathbf{x}}) + \mathbf{\Lambda}_{ij}(\mathbf{x}, \breve{\mathbf{x}}), \tag{41}$$

where

$$\mathbf{\Gamma}_{ijk}(\mathbf{x}, \breve{\mathbf{x}}) = \begin{bmatrix} \hat{\mathbf{\Xi}}_{jk}^{(11)}(\breve{\mathbf{x}}) + \hat{\mathbf{\Xi}}_{jk}^{(11)}(\breve{\mathbf{x}})^T + \alpha_1 \mathbf{Y}^{-1} & \hat{\mathbf{\Xi}}_{ijk}^{(21)}(\mathbf{x}, \breve{\mathbf{x}})^T \\ * & -\alpha_2 \mathbf{I} \end{bmatrix}, \tag{42}$$

$$\mathbf{\Lambda}_{ij}(\mathbf{x}, \breve{\mathbf{x}}) = \begin{bmatrix} -\alpha_1 \mathbf{Y}^{-1} & \hat{\mathbf{\Xi}}_{ij}^{(12)}(\mathbf{x}, \breve{\mathbf{x}}) \\ * & \hat{\mathbf{\Xi}}_{ij}^{(22)}(\mathbf{x}, \breve{\mathbf{x}}) + \alpha_2 \mathbf{I} \end{bmatrix}. \tag{43}$$

Remark 2: The decoupled matrix in (42) is related to the polynomial fuzzy controller gain $\mathbf{G}_k(\breve{\mathbf{x}})$ while the one in (43) is related to the polynomial fuzzy observer gain $\mathbf{L}_j(\breve{\mathbf{x}})$. In this case, more arrangement can be imposed on (43) without affecting (42) which is already a convex problem. Other techniques such as completing squares (Lemma 1 and Lemma 2) [41] and Finsler's lemma [42] can also be used to further separate decision variables instead of matrix decoupling technique [40]. However, they increase the dimension of matrices or increase the number of decision variables resulting in higher computational demand. In contrast, using matrix decoupling technique leads to smaller dimension of matrices or less number of decision variables at the expense of larger number of stability conditions.

Hence, $\dot{V}(\mathbf{z}) < 0$ holds if

$$\sum_{i,j,k=1}^{p} \tilde{w}_{ijk} \Big(\mathbf{\Gamma}_{ijk}(\mathbf{x}, \breve{\mathbf{x}}) + \beta \mathbf{\Phi}^{(13)} (\mathbf{\Phi}^{(13)})^T \Big) < 0, \tag{44}$$

$$\sum_{i,j,k=1}^{p} \tilde{w}_{ijk} \mathbf{\Lambda}_{ij}(\mathbf{x}, \breve{\mathbf{x}}) + \frac{1}{\beta} \Big(\sum_{i,j,k=1}^{p} \tilde{w}_{ijk} \mathbf{\Theta}_{ijk}^{(13)}(\mathbf{x}, \breve{\mathbf{x}}) \Big) \times \Big(\sum_{i,j,k=1}^{p} \tilde{w}_{ijk} \mathbf{\Theta}_{ijk}^{(13)}(\mathbf{x}, \breve{\mathbf{x}}) \Big)^T < 0. \tag{45}$$

Performing congruence transformation to (45) by pre-multiplying and post-multiplying diag$\{\mathbf{Y}, \mathbf{Y}\}$ to both sides, denoting $\mathbf{M}_j(\breve{\mathbf{x}}) = \mathbf{Y}\mathbf{L}_j(\breve{\mathbf{x}})$, and then applying Schur Complement to both (44) and (45), we obtain

$$\sum_{i,j,k=1}^{p} \tilde{w}_{ijk} \mathbf{\Phi}_{ijk}(\mathbf{x}, \breve{\mathbf{x}}) < 0, \tag{46}$$

$$\sum_{i,j,k=1}^{p} \tilde{w}_{ijk} \mathbf{\Theta}_{ijk}(\mathbf{x}, \breve{\mathbf{x}}) < 0, \tag{47}$$

where $\mathbf{\Phi}_{ijk}(\mathbf{x}, \breve{\mathbf{x}})$ and $\mathbf{\Theta}_{ijk}(\mathbf{x}, \breve{\mathbf{x}})$ are defined in (11) and (12), respectively. By grouping terms with same membership functions, $\dot{V}(\mathbf{z}) < 0$ can be achieved by satisfying conditions (9) and (10). The proof is completed. ■

B. Polynomial Fuzzy Controller and Observer with Sampled-Output Measurement

Considering premise variable $f_\eta(\mathbf{x})$ depending on unmeasurable system states $\mathbf{x}$ and output matrix $\mathbf{C}_i$ not depending on system states $\mathbf{x}$, we denote sampled output $\mathbf{y}$ as $\mathbf{y}_s$, where $\mathbf{y}_s = \mathbf{y}(t_{\tilde{s}})$, $t_{\tilde{s}}, \tilde{s} = 1, 2, \ldots, \infty$, is the sampling time and $t_{\tilde{s}+1} - t_{\tilde{s}} \leq h$. The input-delay approach [47] is employed to represent the sampling behavior. Denoting $\tau(t) = t - t_{\tilde{s}} < h$ for $t_{\tilde{s}} \leq t < t_{\tilde{s}+1}$, the sampled output vector can be written as $\mathbf{y}_s = \mathbf{y}(t - \tau(t))$. Similarly, the sampled system state vector can be written as $\mathbf{x}_s = \mathbf{x}(t - \tau(t))$.

Remark 3: In case using sampled-output measurements, the output matrix $\mathbf{C}_i$ does not depend on system states $\mathbf{x}$. If $\mathbf{C}_i(\mathbf{x})$ is considered to be a polynomial matrix of $\mathbf{x}$, $\mathbf{C}_i(\mathbf{x}_s)$ and $\mathbf{C}_i(\breve{\mathbf{x}}_s)$ will exist in the stability analysis which is more difficult to be handled. Therefore, constant output matrix $\mathbf{C}_i$ is considered in this paper to ease the design and analysis.

We apply the following polynomial fuzzy observer to estimate the system states in (1):

$$\begin{aligned} \dot{\breve{\mathbf{x}}} &= \sum_{j=1}^{p} w_j(\breve{\mathbf{x}}) \Big(\mathbf{A}_j(\breve{\mathbf{x}})\breve{\mathbf{x}} + \mathbf{B}_j(\breve{\mathbf{x}})\mathbf{u} + \mathbf{L}_j(\breve{\mathbf{x}})(\mathbf{y}_s - \breve{\mathbf{y}}_s) \Big), \\ \mathbf{y}_s &= \sum_{i=1}^{p} w_i(\mathbf{x}_s) \mathbf{C}_i \mathbf{x}_s, \\ \breve{\mathbf{y}}_s &= \sum_{l=1}^{p} w_l(\breve{\mathbf{x}}_s) \mathbf{C}_l \breve{\mathbf{x}}_s, \end{aligned} \tag{48}$$

where $\breve{\mathbf{x}}_s \in \Re^n$ and $\breve{\mathbf{y}}_s \in \Re^l$ are the estimated sampled system states and output, respectively.

With the PDC design approach [3], [7], the polynomial fuzzy controller is given in (3). Recalling that the estimation error is defined as $\mathbf{e} = \mathbf{x} - \breve{\mathbf{x}}$, we define the sampled estimation error as $\mathbf{e}_s = \mathbf{x}_s - \breve{\mathbf{x}}_s$, and then we have the closed-loop system (shown in Fig. 2) consisting of the polynomial fuzzy model (1), the polynomial fuzzy controller (3) and the polynomial fuzzy observer (48) as follows:

$$\dot{\mathbf{x}} = \sum_{i=1}^{p} \sum_{j=1}^{p} w_i(\mathbf{x}) w_j(\breve{\mathbf{x}}) \Big((\mathbf{A}_i(\mathbf{x}) + \mathbf{B}_i(\mathbf{x})\mathbf{G}_j(\breve{\mathbf{x}}))\breve{\mathbf{x}} + \mathbf{A}_i(\mathbf{x})\mathbf{e} \Big), \tag{49}$$

$$\dot{\breve{\mathbf{x}}} = \sum_{i=1}^{p} \sum_{j=1}^{p} \sum_{k=1}^{p} \sum_{l=1}^{p} w_i(\mathbf{x}_s) w_j(\breve{\mathbf{x}}) w_k(\breve{\mathbf{x}}) w_l(\breve{\mathbf{x}}_s) \Big((\mathbf{A}_j(\breve{\mathbf{x}}) + \mathbf{B}_j(\breve{\mathbf{x}})\mathbf{G}_k(\breve{\mathbf{x}}))\breve{\mathbf{x}} + \mathbf{L}_j(\breve{\mathbf{x}})(\mathbf{C}_i - \mathbf{C}_l)\breve{\mathbf{x}}_s + \mathbf{L}_j(\breve{\mathbf{x}})\mathbf{C}_i\mathbf{e}_s \Big), \tag{50}$$

$$\dot{\mathbf{e}} = \sum_{i=1}^{p} \sum_{j=1}^{p} \sum_{k=1}^{p} \sum_{l=1}^{p} \sum_{m=1}^{p} w_i(\mathbf{x}_s) w_j(\breve{\mathbf{x}}) w_k(\breve{\mathbf{x}}) w_l(\breve{\mathbf{x}}_s) w_m(\mathbf{x}) \Big((\mathbf{A}_m(\mathbf{x}) - \mathbf{A}_j(\breve{\mathbf{x}}) + (\mathbf{B}_m(\mathbf{x}) - \mathbf{B}_j(\breve{\mathbf{x}}))\mathbf{G}_k(\breve{\mathbf{x}}))\breve{\mathbf{x}} + \mathbf{A}_m(\mathbf{x})\mathbf{e} - \mathbf{L}_j(\breve{\mathbf{x}})(\mathbf{C}_i - \mathbf{C}_l)\breve{\mathbf{x}}_s - \mathbf{L}_j(\breve{\mathbf{x}})\mathbf{C}_i\mathbf{e}_s \Big). \tag{51}$$

The control objective is to make the augmented observer-control system ((50) and (51)) asymptotically stable, i.e., $\breve{\mathbf{x}} \to 0$ and $\mathbf{e} \to 0$ as time $t \to \infty$, by determining the polynomial controller gain $\mathbf{G}_k(\breve{\mathbf{x}})$ and polynomial observer gain $\mathbf{L}_j(\breve{\mathbf{x}})$.

Theorem 2: The augmented PFMB observer-control system (formed by (50) and (51)) is guaranteed to be asymptotically stable if there exist matrices $\mathbf{X} \in \Re^{N \times N}, \mathbf{Y} \in$

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2015.2402685, IEEE Transactions on Fuzzy Systems

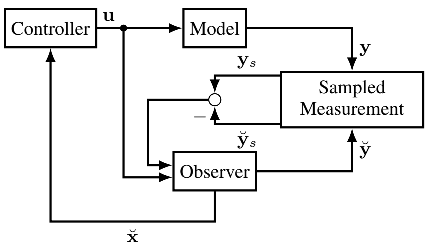


Fig. 2. A block diagram of PFMB observer-control systems with sampled-output measurement.

$\Re^{N\times N}, \tilde{\mathbf{Q}} \in \Re^{2N\times 2N}, \mathbf{N}_k(\breve{\mathbf{x}}) \in \Re^{m\times N}, \mathbf{M}_j(\breve{\mathbf{x}}) \in \Re^{N\times l}, k = 1,2,\ldots,p, j = 1,2,\ldots,p$, and predefined scalers $\alpha_1 > 0, \alpha_2 > 0, \alpha_3 > 0, \alpha_4 > 0, \beta > 0$ and γ such that the following SOS-based conditions are satisfied:

$$\nu^T(\mathbf{X} - \varepsilon_1\mathbf{I})\nu \text{ is SOS;} \tag{52}$$

$$\nu^T(\mathbf{Y} - \varepsilon_2\mathbf{I})\nu \text{ is SOS;} \tag{53}$$

$$\nu^T(\tilde{\mathbf{Q}} - \varepsilon_3\mathbf{I})\nu \text{ is SOS;} \tag{54}$$

$$-\nu^T(\mathbf{\Phi}_{jkm}(\mathbf{x},\breve{\mathbf{x}}) + \mathbf{\Phi}_{kjm}(\mathbf{x},\breve{\mathbf{x}}) + \varepsilon_4(\mathbf{x},\breve{\mathbf{x}})\mathbf{I})\nu \text{ is SOS } \forall m, j \leq k; \tag{55}$$

$$-\nu^T(\mathbf{\Theta}_{ijlm}(\mathbf{x},\breve{\mathbf{x}}) + \varepsilon_5(\mathbf{x},\breve{\mathbf{x}})\mathbf{I})\nu \text{ is SOS } \forall i,j,l,m; \tag{56}$$

where

$$\mathbf{\Phi}_{jkm}(\mathbf{x},\breve{\mathbf{x}}) = \begin{bmatrix} \tilde{\mathbf{\Gamma}}_{jkm}(\mathbf{x},\breve{\mathbf{x}}) & \tilde{\mathbf{\Phi}}^{(12)} & \mathbf{\Phi}^{(13)} \\ * & -\frac{1}{\beta}\mathbf{I} & \mathbf{0} \\ * & * & -\frac{1}{\alpha_2}\mathbf{Y} \end{bmatrix}, \tag{57}$$

$$\mathbf{\Theta}_{ijlm}(\mathbf{x},\breve{\mathbf{x}}) = \begin{bmatrix} \tilde{\mathbf{\Lambda}}_{ijm}(\mathbf{x},\breve{\mathbf{x}}) & \mathbf{\Theta}^{(12)} & \mathbf{\Theta}^{(13)} & \mathbf{\Theta}^{(14)} & \tilde{\mathbf{\Theta}}^{(15)}_{ijl}(\breve{\mathbf{x}}) \\ * & -\frac{1}{\alpha_1}\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & -\frac{1}{\alpha_4}\mathbf{I} & \mathbf{0} & \mathbf{0} \\ * & * & * & -\frac{1}{\alpha_3}\mathbf{I} & \mathbf{0} \\ * & * & * & * & -\beta\mathbf{I} \end{bmatrix}, \tag{58}$$

$$\tilde{\mathbf{\Gamma}}_{jkm}(\mathbf{x},\breve{\mathbf{x}}) = \begin{bmatrix} \mathbf{\Gamma}^{(11)}_{jkm}(\mathbf{x},\breve{\mathbf{x}}) & \mathbf{0} & \mathbf{\Gamma}^{(14)}_{jkm}(\mathbf{x},\breve{\mathbf{x}}) \\ * & -\alpha_4\mathbf{I} & \mathbf{0} \\ * & * & \tilde{\mathbf{\Gamma}}^{(44)} \end{bmatrix}, \tag{59}$$

$$\tilde{\mathbf{\Phi}}^{(12)} = [\mathbf{0}_{N\times 2N} \quad \mathbf{X} \quad \mathbf{0}_{N\times 2N}]^T, \tag{60}$$

$$\mathbf{\Phi}^{(13)} = [\mathbf{0}_{N\times 3N} \quad \mathbf{I} \quad \mathbf{0}_{N\times N}]^T, \tag{61}$$

$$\tilde{\mathbf{\Lambda}}_{ijm}(\mathbf{x},\breve{\mathbf{x}}) = \begin{bmatrix} \tilde{\mathbf{\Lambda}}^{(11)}_m(\mathbf{x}) - \tilde{\mathbf{Q}} & \tilde{\mathbf{\Lambda}}^{(12)}_{ij}(\breve{\mathbf{x}}) + \tilde{\mathbf{Q}} & \mathbf{0} & \tilde{\mathbf{\Lambda}}^{(14)}_m(\mathbf{x}) \\ * & -2\tilde{\mathbf{Q}} & \tilde{\mathbf{Q}} & \tilde{\mathbf{\Lambda}}^{(24)}_{ij}(\breve{\mathbf{x}}) \\ * & * & -\tilde{\mathbf{Q}} & \mathbf{0} \\ * & * & * & \gamma^2\tilde{\mathbf{Q}} + \tilde{\mathbf{\Lambda}}^{(44)} \end{bmatrix}, \tag{62}$$

$$\mathbf{\Theta}^{(12)} = [\mathbf{0}_{N\times N} \quad \mathbf{Y} \quad \mathbf{0}_{N\times 6N}]^T, \tag{63}$$

$$\mathbf{\Theta}^{(13)} = [\mathbf{0}_{N\times 2N} \quad \mathbf{Y} \quad \mathbf{0}_{N\times 5N}]^T, \tag{64}$$

$$\mathbf{\Theta}^{(14)} = [\mathbf{0}_{N\times 7N} \quad \mathbf{Y}]^T, \tag{65}$$

$$\tilde{\mathbf{\Theta}}^{(15)}_{ijl}(\breve{\mathbf{x}}) = [\tilde{\mathbf{H}}_{ijl}(\breve{\mathbf{x}})^T \quad -\tilde{\mathbf{H}}_{ijl}(\breve{\mathbf{x}})^T \quad \mathbf{0}_{N\times 4N} \quad h\tilde{\mathbf{H}}_{ijl}(\breve{\mathbf{x}})^T \quad -h\tilde{\mathbf{H}}_{ijl}(\breve{\mathbf{x}})^T]^T, \tag{66}$$

$$\mathbf{\Gamma}^{(11)}_{jkm}(\mathbf{x},\breve{\mathbf{x}}) = \begin{bmatrix} \hat{\mathbf{\Xi}}^{(11)}_{jk}(\breve{\mathbf{x}}) + \hat{\mathbf{\Xi}}^{(11)}_{jk}(\breve{\mathbf{x}})^T & \hat{\mathbf{\Xi}}^{(21)}_{jkm}(\mathbf{x},\breve{\mathbf{x}})^T \\ * & -\alpha_1\mathbf{I} \end{bmatrix}, \tag{67}$$

$$\mathbf{\Gamma}^{(14)}_{jkm}(\mathbf{x},\breve{\mathbf{x}}) = \begin{bmatrix} h\hat{\mathbf{\Xi}}^{(11)}_{jk}(\breve{\mathbf{x}})^T & h\hat{\mathbf{\Xi}}^{(21)}_{jkm}(\mathbf{x},\breve{\mathbf{x}})^T \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \tag{68}$$

$$\tilde{\mathbf{\Gamma}}^{(44)} = \begin{bmatrix} -2\gamma\mathbf{X} & \mathbf{0} \\ \mathbf{0} & -\alpha_3\mathbf{I} \end{bmatrix}, \tag{69}$$

$$\tilde{\mathbf{\Lambda}}^{(11)}_m(\mathbf{x}) = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{\Xi}}^{(22)}_m(\mathbf{x}) + \tilde{\mathbf{\Xi}}^{(22)}_m(\mathbf{x})^T \end{bmatrix}, \tag{70}$$

$$\tilde{\mathbf{\Lambda}}^{(12)}_{ij}(\breve{\mathbf{x}}) = \begin{bmatrix} \mathbf{0} & \tilde{\mathbf{K}}_{ij}(\breve{\mathbf{x}}) \\ \mathbf{0} & -\tilde{\mathbf{K}}_{ij}(\breve{\mathbf{x}}) \end{bmatrix}, \tag{71}$$

$$\tilde{\mathbf{\Lambda}}^{(14)}_m(\mathbf{x}) = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & h\tilde{\mathbf{\Xi}}^{(22)}_m(\mathbf{x})^T \end{bmatrix}, \tag{72}$$

$$\tilde{\mathbf{\Lambda}}^{(24)}_{ij}(\breve{\mathbf{x}}) = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ h\tilde{\mathbf{K}}_{ij}(\breve{\mathbf{x}})^T & -h\tilde{\mathbf{K}}_{ij}(\breve{\mathbf{x}})^T \end{bmatrix}, \tag{73}$$

$$\tilde{\mathbf{\Lambda}}^{(44)} = \begin{bmatrix} -\alpha_2\mathbf{Y} & \mathbf{0} \\ \mathbf{0} & -2\gamma\mathbf{Y} \end{bmatrix}, \tag{74}$$

$$\hat{\mathbf{\Xi}}^{(11)}_{jk}(\breve{\mathbf{x}}) = \mathbf{A}_j(\breve{\mathbf{x}})\mathbf{X} + \mathbf{B}_j(\breve{\mathbf{x}})\mathbf{N}_k(\breve{\mathbf{x}}), \tag{75}$$

$$\hat{\mathbf{\Xi}}^{(21)}_{jkm}(\mathbf{x},\breve{\mathbf{x}}) = (\mathbf{A}_m(\mathbf{x}) - \mathbf{A}_j(\breve{\mathbf{x}}))\mathbf{X} + (\mathbf{B}_m(\mathbf{x}) - \mathbf{B}_j(\breve{\mathbf{x}}))\mathbf{N}_k(\breve{\mathbf{x}}), \tag{76}$$

$$\tilde{\mathbf{\Xi}}^{(22)}_m(\mathbf{x}) = \mathbf{Y}\mathbf{A}_m(\mathbf{x}), \tag{77}$$

$$\tilde{\mathbf{K}}_{ij}(\breve{\mathbf{x}}) = \mathbf{M}_j(\breve{\mathbf{x}})\mathbf{C}_i, \tag{78}$$

$$\tilde{\mathbf{H}}_{ijl}(\breve{\mathbf{x}}) = \mathbf{M}_j(\breve{\mathbf{x}})(\mathbf{C}_i - \mathbf{C}_l); \tag{79}$$

ν is an arbitrary vector independent of $\mathbf{x}$ with appropriate dimensions; $\varepsilon_1 > 0, \varepsilon_2 > 0, \varepsilon_3 > 0$, $\varepsilon_4(\mathbf{x},\breve{\mathbf{x}}) > 0$ and $\varepsilon_5(\mathbf{x},\breve{\mathbf{x}}) > 0$ are predefined scalar polynomials; and the polynomial controller and polynomial observer gains are given by $\mathbf{G}_k(\breve{\mathbf{x}}) = \mathbf{N}_k(\breve{\mathbf{x}})\mathbf{X}^{-1}$ and $\mathbf{L}_j(\breve{\mathbf{x}}) = \mathbf{Y}^{-1}\mathbf{M}_j(\breve{\mathbf{x}})$, respectively.

Proof: Defining the augmented vectors $\mathbf{z} = [\breve{\mathbf{x}}^T \quad \mathbf{e}^T]^T, \mathbf{z}_s = [\breve{\mathbf{x}}_s^T \quad \mathbf{e}_s^T]^T$, and the summation term $\sum_{i,j,k,l,m=1}^{p} \tilde{w}_{ijklm} = \sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p}\sum_{l=1}^{p}\sum_{m=1}^{p} w_i(\mathbf{x}_s)w_j(\breve{\mathbf{x}})w_k(\breve{\mathbf{x}})w_l(\breve{\mathbf{x}}_s)w_m(\mathbf{x})$, the augmented system becomes

$$\dot{\mathbf{z}} = \sum_{i,j,k,l,m=1}^{p} \tilde{w}_{ijklm}\big(\hat{\mathbf{A}}_{jkm}(\mathbf{x},\breve{\mathbf{x}})\mathbf{z} + \hat{\mathbf{B}}_{ijl}(\breve{\mathbf{x}})\mathbf{z}_s\big), \tag{80}$$

where

$$\hat{\mathbf{A}}_{jkm}(\mathbf{x},\breve{\mathbf{x}}) = \begin{bmatrix} \mathbf{\Xi}^{(11)}_{jk}(\breve{\mathbf{x}}) & \mathbf{0} \\ \mathbf{\Xi}^{(21)}_{jkm}(\mathbf{x},\breve{\mathbf{x}}) & \mathbf{\Xi}^{(22)}_m(\mathbf{x}) \end{bmatrix}, \tag{81}$$

$$\hat{\mathbf{B}}_{ijl}(\breve{\mathbf{x}}) = \begin{bmatrix} \mathbf{H}_{ijl}(\breve{\mathbf{x}}) & \mathbf{K}_{ij}(\breve{\mathbf{x}}) \\ -\mathbf{H}_{ijl}(\breve{\mathbf{x}}) & -\mathbf{K}_{ij}(\breve{\mathbf{x}}) \end{bmatrix}, \tag{82}$$

$$\mathbf{\Xi}^{(11)}_{jk}(\breve{\mathbf{x}}) = \mathbf{A}_j(\breve{\mathbf{x}}) + \mathbf{B}_j(\breve{\mathbf{x}})\mathbf{G}_k(\breve{\mathbf{x}}), \tag{83}$$

$$\mathbf{\Xi}^{(21)}_{jkm}(\mathbf{x},\breve{\mathbf{x}}) = \mathbf{A}_m(\mathbf{x}) - \mathbf{A}_j(\breve{\mathbf{x}}) + (\mathbf{B}_m(\mathbf{x}) - \mathbf{B}_j(\breve{\mathbf{x}}))\mathbf{G}_k(\breve{\mathbf{x}}), \tag{84}$$

$$\mathbf{\Xi}^{(22)}_m(\mathbf{x}) = \mathbf{A}_m(\mathbf{x}), \tag{85}$$

$$\mathbf{K}_{ij}(\breve{\mathbf{x}}) = \mathbf{L}_j(\breve{\mathbf{x}})\mathbf{C}_i, \tag{86}$$

$$\mathbf{H}_{ijl}(\breve{\mathbf{x}}) = \mathbf{L}_j(\breve{\mathbf{x}})(\mathbf{C}_i - \mathbf{C}_l). \tag{87}$$

The following Lyapunov function candidate is employed to investigate the stability of the augmented PFMB observer-

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2015.2402685, IEEE Transactions on Fuzzy Systems

control system with sampled-output measurements (80):

$$V(\mathbf{z})=\mathbf{z}^T\mathbf{P}\mathbf{z}+h\int_{-h}^{0}\int_{t+\sigma}^{t}\dot{\mathbf{z}}(\varphi)^T\mathbf{Q}\dot{\mathbf{z}}(\varphi)d\varphi d\sigma, \quad (88)$$

where $\mathbf{Q}>0, \mathbf{P}=\begin{bmatrix}\mathbf{X}^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{Y}\end{bmatrix}, \mathbf{X}>0, \mathbf{Y}>0$, and thus $\mathbf{P}>0$. The time derivative of $V(\mathbf{z})$ is obtained as follows:

$$\dot{V}(\mathbf{z})=\dot{\mathbf{z}}^T\mathbf{P}\mathbf{z}+\mathbf{z}^T\mathbf{P}\dot{\mathbf{z}}+h^2\dot{\mathbf{z}}^T\mathbf{Q}\dot{\mathbf{z}}-h\int_{t-h}^{t}\dot{\mathbf{z}}(\varphi)^T\mathbf{Q}\mathbf{z}(\varphi)d\varphi. \quad (89)$$

Denoting augmented vectors $\mathbf{z}_h=[\breve{\mathbf{x}}(t-h)^T \quad \mathbf{e}(t-h)^T]^T, \mathbf{Z}=[\mathbf{z}^T \quad \mathbf{z}_s^T \quad \mathbf{z}_h^T]^T$, and using Lemma 3, we obtain

$$\begin{aligned}&-h\int_{t-h}^{t}\dot{\mathbf{z}}(\varphi)^T\mathbf{Q}\dot{\mathbf{z}}(\varphi)d\varphi\\ \le&-(\mathbf{z}-\mathbf{z}_s)^T\mathbf{Q}(\mathbf{z}-\mathbf{z}_s)-(\mathbf{z}_s-\mathbf{z}_h)^T\mathbf{Q}(\mathbf{z}_s-\mathbf{z}_h). \quad (90)\end{aligned}$$

Then $\dot{V}(\mathbf{z})$ becomes

$$\begin{aligned}\dot{V}(\mathbf{z})\le\mathbf{Z}^T\Big(&\sum_{i,j,k,l,m=1}^{p}\tilde{w}_{ijklm}\mathbf{\Upsilon}_{ijklm}^{(11)}(\mathbf{x},\breve{\mathbf{x}})\\ &+(\sum_{i,j,k,l,m=1}^{p}\tilde{w}_{ijklm}\mathbf{\Upsilon}_{ijklm}^{(12)}(\mathbf{x},\breve{\mathbf{x}}))\mathbf{P}^{-1}\mathbf{Q}\mathbf{P}^{-1}\\ &\times(\sum_{i,j,k,l,m=1}^{p}\tilde{w}_{ijklm}\mathbf{\Upsilon}_{ijklm}^{(12)}(\mathbf{x},\breve{\mathbf{x}}))^T\Big)\mathbf{Z}, \quad (91)\end{aligned}$$

where

$$\mathbf{\Upsilon}_{ijklm}^{(11)}(\mathbf{x},\breve{\mathbf{x}})=\begin{bmatrix}\mathbf{\Omega}(\mathbf{x},\breve{\mathbf{x}}) & \mathbf{P}\hat{\mathbf{B}}_{ijl}(\mathbf{x},\breve{\mathbf{x}})+\mathbf{Q} & \mathbf{0}\\ * & -2\mathbf{Q} & \mathbf{Q}\\ * & * & -\mathbf{Q}\end{bmatrix}, \quad (92)$$

$$\mathbf{\Omega}(\mathbf{x},\breve{\mathbf{x}})=\mathbf{P}\hat{\mathbf{A}}_{jkm}(\mathbf{x},\breve{\mathbf{x}})+\hat{\mathbf{A}}_{jkm}(\mathbf{x},\breve{\mathbf{x}})^T\mathbf{P}-\mathbf{Q}, \quad (93)$$

$$\mathbf{\Upsilon}_{ijklm}^{(12)}(\mathbf{x},\breve{\mathbf{x}})=[h\mathbf{P}\hat{\mathbf{A}}_{jkm}(\mathbf{x},\breve{\mathbf{x}}) \quad h\mathbf{P}\hat{\mathbf{B}}_{ijl}(\mathbf{x},\breve{\mathbf{x}}) \quad \mathbf{0}]^T. \quad (94)$$

Using Schur Complement, $\dot{V}(\mathbf{z})<0$ holds if

$$\sum_{i,j,k,l,m=1}^{p}\tilde{w}_{ijklm}\mathbf{\Upsilon}_{ijklm}(\mathbf{x},\breve{\mathbf{x}})<0, \quad (95)$$

where

$$\mathbf{\Upsilon}_{ijklm}(\mathbf{x},\breve{\mathbf{x}})=\begin{bmatrix}\mathbf{\Upsilon}_{ijklm}^{(11)}(\mathbf{x},\breve{\mathbf{x}}) & \mathbf{\Upsilon}_{ijklm}^{(12)}(\mathbf{x},\breve{\mathbf{x}})\\ * & -\mathbf{P}\mathbf{Q}^{-1}\mathbf{P}\end{bmatrix}. \quad (96)$$

Applying Lemma 2 to the term $-\mathbf{P}\mathbf{Q}^{-1}\mathbf{P}$ and then performing congruence transformation to (95) by pre-multiplying and post-multiplying diag$\{\mathbf{P}^{-1},\mathbf{P}^{-1},\mathbf{P}^{-1},\mathbf{P}^{-1}\}$, we have

$$\sum_{i,j,k,l,m=1}^{p}\tilde{w}_{ijklm}\hat{\mathbf{\Upsilon}}_{ijklm}(\mathbf{x},\breve{\mathbf{x}})<0, \quad (97)$$

where

$$\hat{\mathbf{\Upsilon}}_{ijklm}(\mathbf{x},\breve{\mathbf{x}})=\begin{bmatrix}\hat{\mathbf{\Upsilon}}_{jkm}^{(11)}(\mathbf{x},\breve{\mathbf{x}})-\hat{\mathbf{Q}} & \hat{\mathbf{\Upsilon}}_{ijl}^{(12)}(\breve{\mathbf{x}})+\hat{\mathbf{Q}} & \mathbf{0} & \hat{\mathbf{\Upsilon}}_{jkm}^{(14)}(\mathbf{x},\breve{\mathbf{x}})\\ * & -2\hat{\mathbf{Q}} & \hat{\mathbf{Q}} & \hat{\mathbf{\Upsilon}}_{ijl}^{(24)}(\breve{\mathbf{x}})\\ * & * & -\hat{\mathbf{Q}} & \mathbf{0}\\ * & * & * & \gamma^2\hat{\mathbf{Q}}+\hat{\mathbf{\Upsilon}}^{(44)}\end{bmatrix}, \quad (98)$$

$$\hat{\mathbf{\Upsilon}}_{jkm}^{(11)}(\mathbf{x},\breve{\mathbf{x}})=\begin{bmatrix}\hat{\mathbf{\Xi}}_{jk}^{(11)}(\breve{\mathbf{x}})+\hat{\mathbf{\Xi}}_{jk}^{(11)}(\breve{\mathbf{x}})^T & \hat{\mathbf{\Xi}}_{jkm}^{(21)}(\mathbf{x},\breve{\mathbf{x}})^T\\ * & \hat{\mathbf{\Xi}}_{m}^{(22)}(\mathbf{x})+\hat{\mathbf{\Xi}}_{m}^{(22)}(\mathbf{x})^T\end{bmatrix}, \quad (99)$$

$$\hat{\mathbf{\Upsilon}}_{ijl}^{(12)}(\breve{\mathbf{x}})=\begin{bmatrix}\mathbf{H}_{ijl}(\breve{\mathbf{x}})\mathbf{X} & \hat{\mathbf{K}}_{ij}(\breve{\mathbf{x}})\\ -\mathbf{H}_{ijl}(\breve{\mathbf{x}})\mathbf{X} & -\hat{\mathbf{K}}_{ij}(\breve{\mathbf{x}})\end{bmatrix}, \quad (100)$$

$$\hat{\mathbf{\Upsilon}}_{jkm}^{(14)}(\mathbf{x},\breve{\mathbf{x}})=\begin{bmatrix}h\hat{\mathbf{\Xi}}_{jk}^{(11)}(\breve{\mathbf{x}})^T & h\hat{\mathbf{\Xi}}_{jkm}^{(21)}(\mathbf{x},\breve{\mathbf{x}})^T\\ \mathbf{0} & h\hat{\mathbf{\Xi}}_{m}^{(22)}(\mathbf{x})^T\end{bmatrix}, \quad (101)$$

$$\hat{\mathbf{\Upsilon}}_{ijl}^{(24)}(\breve{\mathbf{x}})=\begin{bmatrix}h\mathbf{X}\mathbf{H}_{ijl}(\breve{\mathbf{x}})^T & -h\mathbf{X}\mathbf{H}_{ijl}(\breve{\mathbf{x}})^T\\ h\hat{\mathbf{K}}_{ij}(\breve{\mathbf{x}})^T & -h\hat{\mathbf{K}}_{ij}(\breve{\mathbf{x}})^T\end{bmatrix}, \quad (102)$$

$$\hat{\mathbf{\Upsilon}}^{(44)}=\begin{bmatrix}-2\gamma\mathbf{X} & \mathbf{0}\\ \mathbf{0} & -2\gamma\mathbf{Y}^{-1}\end{bmatrix}, \quad (103)$$

$$\hat{\mathbf{\Xi}}_{m}^{(22)}(\mathbf{x})=\mathbf{A}_m(\mathbf{x})\mathbf{Y}^{-1}, \quad (104)$$

$$\hat{\mathbf{K}}_{ij}(\breve{\mathbf{x}})=\mathbf{L}_j(\breve{\mathbf{x}})\mathbf{C}_i\mathbf{Y}^{-1}, \quad (105)$$

$$\hat{\mathbf{Q}}=\mathbf{P}^{-1}\mathbf{Q}\mathbf{P}^{-1}, \quad (106)$$

$\hat{\mathbf{\Xi}}_{jk}^{(11)}(\breve{\mathbf{x}})$ and $\hat{\mathbf{\Xi}}_{jkm}^{(21)}(\mathbf{x},\breve{\mathbf{x}})$ are defined in (75) and (76), respectively.

Similar to the development in Subsection III-A, applying Lemma 1 and matrix decoupling technique [40] to further separate decision variables, we get

$$\begin{aligned}&\sum_{i,j,k,l,m=1}^{p}\tilde{w}_{ijklm}\hat{\mathbf{\Upsilon}}_{ijklm}(\mathbf{x},\breve{\mathbf{x}})\\ \le&\sum_{i,j,k,l,m=1}^{p}\tilde{w}_{ijklm}\Big(\mathbf{\Gamma}_{jkm}(\mathbf{x},\breve{\mathbf{x}})+\mathbf{\Lambda}_{ijm}(\mathbf{x},\breve{\mathbf{x}})\\ &+\beta\mathbf{\Phi}^{(12)}(\mathbf{\Phi}^{(12)})^T\Big)+\frac{1}{\beta}\Big(\sum_{i,j,k,l,m=1}^{p}\tilde{w}_{ijklm}\mathbf{\Theta}_{ijl}^{(15)}(\breve{\mathbf{x}})\Big)\\ &\times\Big(\sum_{i,j,k,l,m=1}^{p}\tilde{w}_{ijklm}\mathbf{\Theta}_{ijl}^{(15)}(\breve{\mathbf{x}})\Big)^T, \quad (107)\end{aligned}$$

where

$$\mathbf{\Phi}^{(12)}=[\mathbf{0}_{N\times 2N} \quad \mathbf{X} \quad \mathbf{0}_{N\times 5N}]^T, \quad (108)$$

$$\begin{aligned}\mathbf{\Theta}_{ijl}^{(15)}(\breve{\mathbf{x}})=[&\mathbf{H}_{ijl}(\breve{\mathbf{x}})^T \quad -\mathbf{H}_{ijl}(\breve{\mathbf{x}})^T \quad \mathbf{0}_{N\times 4N}\\ &h\mathbf{H}_{ijl}(\breve{\mathbf{x}})^T \quad -h\mathbf{H}_{ijl}(\breve{\mathbf{x}})^T]^T, \quad (109)\end{aligned}$$

$$\mathbf{\Gamma}_{jkm}(\mathbf{x},\breve{\mathbf{x}})=\begin{bmatrix}\mathbf{\Gamma}_{jkm}^{(11)}(\mathbf{x},\breve{\mathbf{x}}) & \mathbf{0} & \mathbf{0} & \mathbf{\Gamma}_{jkm}^{(14)}(\mathbf{x},\breve{\mathbf{x}})\\ * & \mathbf{\Gamma}^{(22)} & \mathbf{0} & \mathbf{0}\\ * & * & \mathbf{0} & \mathbf{0}\\ * & * & * & \mathbf{\Gamma}^{(44)}\end{bmatrix}, \quad (110)$$

$$\mathbf{\Gamma}^{(22)}=\begin{bmatrix}-\alpha_4\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}, \quad (111)$$

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2015.2402685, IEEE Transactions on Fuzzy Systems

$$\mathbf{\Gamma}^{(44)} = \begin{bmatrix} -2\gamma\mathbf{X} + \alpha_2\mathbf{Y}^{-1} & \mathbf{0} \\ \mathbf{0} & -\alpha_3\mathbf{I} \end{bmatrix}, \tag{112}$$

$$\mathbf{\Lambda}_{ijm}(\mathbf{x}, \breve{\mathbf{x}}) = \begin{bmatrix} \mathbf{\Lambda}_m^{(11)}(\mathbf{x}) - \hat{\mathbf{Q}} & \mathbf{\Lambda}_{ij}^{(12)}(\breve{\mathbf{x}}) + \hat{\mathbf{Q}} & \mathbf{0} & \mathbf{\Lambda}_m^{(14)}(\mathbf{x}) \\ * & \mathbf{\Lambda}^{(22)} - 2\hat{\mathbf{Q}} & \hat{\mathbf{Q}} & \mathbf{\Lambda}_{ij}^{(24)}(\breve{\mathbf{x}}) \\ * & * & -\hat{\mathbf{Q}} & \mathbf{0} \\ * & * & * & \gamma^2\hat{\mathbf{Q}} + \mathbf{\Lambda}^{(44)} \end{bmatrix}, \tag{113}$$

$$\mathbf{\Lambda}_m^{(11)}(\mathbf{x}) = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{\Xi}}_m^{(22)}(\mathbf{x}) + \hat{\mathbf{\Xi}}_m^{(22)}(\mathbf{x})^T + \alpha_1\mathbf{I} \end{bmatrix}, \tag{114}$$

$$\mathbf{\Lambda}_{ij}^{(12)}(\breve{\mathbf{x}}) = \begin{bmatrix} \mathbf{0} & \hat{\mathbf{K}}_{ij}(\breve{\mathbf{x}}) \\ \mathbf{0} & -\hat{\mathbf{K}}_{ij}(\breve{\mathbf{x}}) \end{bmatrix}, \tag{115}$$

$$\mathbf{\Lambda}_m^{(14)}(\mathbf{x}) = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & h\hat{\mathbf{\Xi}}_m^{(22)}(\mathbf{x})^T \end{bmatrix}, \tag{116}$$

$$\mathbf{\Lambda}^{(22)} = \begin{bmatrix} \alpha_4\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \tag{117}$$

$$\mathbf{\Lambda}_{ij}^{(24)}(\breve{\mathbf{x}}) = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ h\hat{\mathbf{K}}_{ij}(\breve{\mathbf{x}})^T & -h\hat{\mathbf{K}}_{ij}(\breve{\mathbf{x}})^T \end{bmatrix}, \tag{118}$$

$$\mathbf{\Lambda}^{(44)} = \begin{bmatrix} -\alpha_2\mathbf{Y}^{-1} & \mathbf{0} \\ \mathbf{0} & -2\gamma\mathbf{Y}^{-1} + \alpha_3\mathbf{I} \end{bmatrix}, \tag{119}$$

$\mathbf{\Gamma}_{jkm}^{(11)}(\mathbf{x}, \breve{\mathbf{x}}), \mathbf{\Gamma}_{jkm}^{(14)}(\mathbf{x}, \breve{\mathbf{x}})$ are defined in (67) and (68), respectively.

Denoting the summation terms $\sum_{j,k,m=1}^{p} \tilde{w}_{jkm} = \sum_{j=1}^{p}\sum_{k=1}^{p}\sum_{m=1}^{p} w_j(\breve{\mathbf{x}})w_k(\breve{\mathbf{x}})w_m(\mathbf{x})$ and $\sum_{i,j,l,m=1}^{p} \tilde{w}_{ijlm} = \sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{l=1}^{p}\sum_{m=1}^{p} w_i(\mathbf{x}_s)w_j(\breve{\mathbf{x}})w_l(\breve{\mathbf{x}}_s)w_m(\mathbf{x})$, then $\dot{V}(\mathbf{z}) < 0$ holds if

$$\sum_{j,k,m=1}^{p} \tilde{w}_{jkm}\Big(\mathbf{\Gamma}_{jkm}(\mathbf{x}, \breve{\mathbf{x}}) + \beta\mathbf{\Phi}^{(12)}(\mathbf{\Phi}^{(12)})^T\Big) < 0, \tag{120}$$

$$\sum_{i,j,l,m=1}^{p} \tilde{w}_{ijlm}\mathbf{\Lambda}_{ijm}(\mathbf{x}, \breve{\mathbf{x}}) + \frac{1}{\beta}\Big(\sum_{i,j,l,m=1}^{p} \tilde{w}_{ijlm}\mathbf{\Theta}_{ijl}^{(15)}(\breve{\mathbf{x}})\Big) \times \Big(\sum_{i,j,l,m=1}^{p} \tilde{w}_{ijlm}\mathbf{\Theta}_{ijl}^{(15)}(\breve{\mathbf{x}})\Big)^T < 0. \tag{121}$$

Performing congruence transformation to (121) by pre-multiplying and post-multiplying $\text{diag}\{\mathbf{Y}, \mathbf{Y}, \mathbf{Y}, \mathbf{Y}, \mathbf{Y}, \mathbf{Y}, \mathbf{Y}, \mathbf{Y}\}$ to both sides and applying Schur Complement to both (120) and (121), we obtain

$$\sum_{j,k,m=1}^{p} \tilde{w}_{jkm}\mathbf{\Phi}_{jkm}(\mathbf{x}, \breve{\mathbf{x}}) < 0, \tag{122}$$

$$\sum_{i,j,l,m=1}^{p} \tilde{w}_{ijlm}\mathbf{\Theta}_{ijlm}(\mathbf{x}, \breve{\mathbf{x}}) < 0, \tag{123}$$

where $\tilde{\mathbf{Q}} = \begin{bmatrix} \mathbf{Y} & \mathbf{0} \\ \mathbf{0} & \mathbf{Y} \end{bmatrix} \hat{\mathbf{Q}} \begin{bmatrix} \mathbf{Y} & \mathbf{0} \\ \mathbf{0} & \mathbf{Y} \end{bmatrix}$, $\mathbf{\Phi}_{jkm}(\mathbf{x}, \breve{\mathbf{x}})$ and $\mathbf{\Theta}_{ijlm}(\mathbf{x}, \breve{\mathbf{x}})$ are defined in (57) and (58), respectively. By grouping terms with same membership functions, $\dot{V}(\mathbf{z}) < 0$ if conditions (55) and (56) hold. The proof is completed. ■

IV. Simulation Examples

In this section, three simulation examples are provided to validate the proposed stability conditions. In the first example, we consider the stabilization control problem for an inverted pendulum using the proposed PFMB observer-controller. In the second example, sampled-output measurements are considered for the same control problem. In the third example, a nonlinear mass-spring-damper system is also stabilized by the designed PFMB observer-controller.

A. Inverted Pendulum

In this example, we consider an inverted pendulum on a cart [7] in the following state space form:

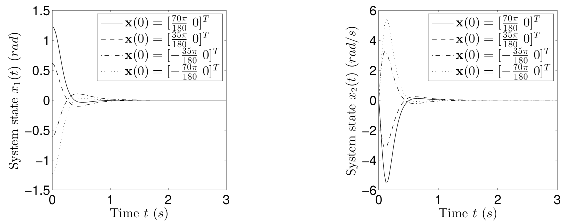
$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \frac{g\sin(x_1) - am_pLx_2^2\sin(x_1)\cos(x_1)}{4L/3 - am_pL\cos^2(x_1)} \\ &\quad - \frac{a\cos(x_1)u}{4L/3 - am_pL\cos^2(x_1)}, \end{aligned} \tag{124}$$

where $\mathbf{x} = [x_1 \quad x_2]^T$ is the state vector; $g = 9.8m/s^2$ is the acceleration of gravity; $m_p = 2kg$ and $M_c = 8kg$ are the mass of the pendulum and the cart, respectively; $a = 1/(m_p + M_c)$; $2L = 1m$ is the length of the pendulum; and $u(t)$ is the control input force imposed on the cart.

Defining the region of interest as $x_1 \in [-\frac{70\pi}{180}, \frac{70\pi}{180}]$, the nonlinear term $f_1(x_1) = \frac{\cos(x_1)}{4L/3 - am_pL\cos^2(x_1)}$ is represented by sector nonlinearity technique [6] as follows: $f_1(x_1) = \mu_{M_1^1}(x_1)f_{1_{min}} + \mu_{M_1^2}(x_1)f_{1_{max}}$, where $\mu_{M_1^1}(x_1) = \frac{f_1(x_1) - f_{1_{max}}}{f_{1_{min}} - f_{1_{max}}}$, $\mu_{M_1^2}(x_1) = 1 - \mu_{M_1^1}(x_1)$, $f_{1_{min}} = 0.5222$, $f_{1_{max}} = 1.7647$. To reduce computational burden, other nonlinear terms $\sin(x_1)$ and $\tan(x_1)$ are approximated by polynomials: $\sin(x_1) \approx s_1x_1$ and $\tan(x_1) \approx t_1x_1$, where $s_1 = 0.8578$ and $t_1 = 1.5534$. As a result, the inverted pendulum is described by a 2-rule polynomial fuzzy model. The system and input matrices in each rule are given by $\mathbf{A}_1(x_2) = \begin{bmatrix} 0 & 1 \\ a_1(x_2) & 0 \end{bmatrix}$, $\mathbf{A}_2(x_2) = \begin{bmatrix} 0 & 1 \\ a_2(x_2) & 0 \end{bmatrix}$, $\mathbf{B}_1 = [0 \quad -f_{1_{min}}a]^T$, and $\mathbf{B}_2 = [0 \quad -f_{1_{max}}a]^T$, where $a_1(x_2) = f_{1_{min}}(gt_1 - am_pLx_2^2s_1)$, $a_2(x_2) = f_{1_{max}}(gt_1 - am_pLx_2^2s_1)$. The measurement of output provided by sensors may be affected by some physical influence such as the angular velocity of the inverted pendulum. Therefore, similar to the example in [40], we suppose the output is a function of system states: $y = x_1 + 0.01x_1x_2$. Then the output matrices are $\mathbf{C}_1(x_2) = \mathbf{C}_2(x_2) = [1 + 0.01x_2 \quad 0]$. The membership functions are $w_1(x_1) = \mu_{M_1^1}(x_1)$ and $w_2(x_1) = \mu_{M_1^2}(x_1)$. It is assumed that both system states x_1 and x_2 are unmeasurable.

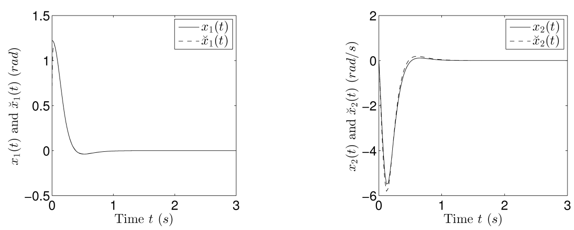
It can be seen that the premise variable $f_1(x_1)$ and the output matrix $\mathbf{C}_i(x_2)$ all depend on unmeasurable system states x_1 or x_2, and thus Theorem 1 is employed to obtain a PFMB observer-controller to stabilize the inverted pendulum. We choose $\alpha_1 = 1 \times 10^3, \alpha_2 = 1 \times 10^6, \beta = 1 \times 10^{-2}$, $\mathbf{N}_k(\breve{x}_2)$ of degree 0 and 2, $\mathbf{M}_j(\breve{x}_2)$ of degree 0 and 1, $\varepsilon_1 = \varepsilon_2 = 1 \times 10^{-3}$, and $\varepsilon_3 = \varepsilon_4 = 1 \times 10^{-7}$. The polynomial controller gains are obtained as $\mathbf{G}_1(\breve{x}_2) = [-1.1623 \times 10^{-2}\breve{x}_2^2 + 1.5144 \times 10^3 \quad 2.5661 \times 10^{-2}\breve{x}_2^2 + 1.6857 \times 10^2]$ and $\mathbf{G}_2(\breve{x}_2) = [-1.2124 \times 10^{-1}\breve{x}_2^2 + 7.7898 \times 10^2 \quad 2.7568 \times 10^{-2}\breve{x}_2^2 + 1.0284 \times 10^2]$, and the polynomial observer gains are obtained as $\mathbf{L}_1(\breve{x}_2) = [-6.0760 \times 10^{-2}\breve{x}_2 + 1.1223 \times 10^2 \quad -$

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2015.2402685, IEEE Transactions on Fuzzy Systems



(a) Time response of $x_1(t)$. (b) Time response of $x_2(t)$.

Fig. 3. Time response of system states of the inverted pendulum with 4 different initial conditions.



(a) Time response of $x_1(t)$ and $\breve{x}_1(t)$. (b) Time response of $x_2(t)$ and $\breve{x}_2(t)$.

Fig. 4. Time response of system states and estimated states for $\mathbf{x}(0) = [\frac{70\pi}{180} \quad 0]^T$.

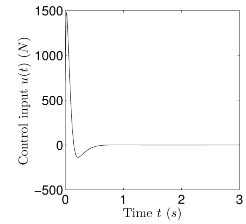
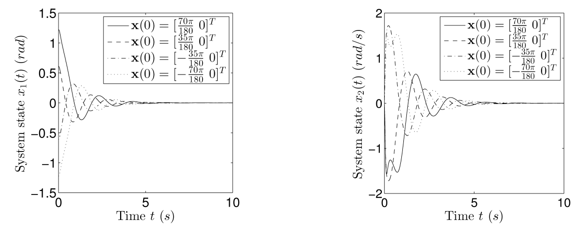


Fig. 5. Time response of control input $u(t)$ for $\mathbf{x}(0) = [\frac{70\pi}{180} \quad 0]^T$.



(a) Time response of $x_1(t)$. (b) Time response of $x_2(t)$.

Fig. 6. Time response of system states of the inverted pendulum with 4 different initial conditions.

$3.5682\times 10^{-2}\breve{x}_2 + 1.2580\times 10^2]^T$ and $\mathbf{L}_2(\breve{x}_2) = [-6.0760\times 10^{-2}\breve{x}_2 + 1.1223\times 10^2 \quad -3.5682\times 10^{-2}\breve{x}_2 + 1.2580\times 10^2]^T$.

We apply the above polynomial controller gains and polynomial observer gains to the original dynamic system of the inverted pendulum (124). Considering 4 different initial conditions, the inverted pendulum is successfully stabilized where the time response of system states are shown in Fig. 3. To demonstrate the estimated system states offered by the polynomial fuzzy observer, we choose one of the above initiation conditions $\mathbf{x}(0) = [\frac{70\pi}{180} \quad 0]^T$ and $\breve{\mathbf{x}}(0) = [\frac{35\pi}{180} \quad 0]^T$ for demonstration purposes and the estimated system states are shown in Fig. 4. The corresponding control input is shown in Fig. 5. It can be seen that the proposed polynomial fuzzy observer is an effective tool for nonlinear systems to observe unmeasurable states.

B. Inverted Pendulum with Sampled-Output Measurements

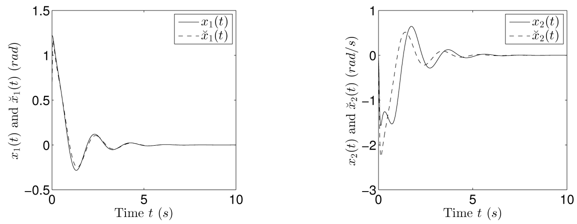
In this example, we consider the same inverted pendulum in (124). In addition, sampled-output measurements are employed for the design of PFMB observer-controller where the sampling interval is chosen to be $h = 0.05$ seconds. The output is assumed to be a function of system states: $y = -0.161 f_1(x_1) + 1.1841$. Consequently, the output matrices are $\mathbf{C}_1 = [1.1 \quad 0]$ and $\mathbf{C}_2 = [0.9 \quad 0]$. The membership functions are the same as the first example.

Theorem 2 is employed for the design of PFMB observer-controller. We choose $\alpha_1 = 1\times 10^6, \alpha_2 = 1\times 10^5, \alpha_3 = 1\times 10^3, \alpha_4 = 1\times 10^3, \beta = 1\times 10^{-2}, \gamma = 1\times 10^{-1}$, $\mathbf{N}_k(\breve{x}_2)$ of degree 0 and 2, $\mathbf{M}_j(\breve{x}_2)$ of degree 0 and 2, $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1\times 10^{-3}$, and $\varepsilon_4 = \varepsilon_5 = 1\times 10^{-7}$. The polynomial controller gains are obtained as $\mathbf{G}_1(\breve{x}_2) = [-2.1745\times 10^{-1}\breve{x}_2^2 + 1.1463\times 10^3 \quad 3.8649\times 10^{-2}\breve{x}_2^2 + 4.6925\times 10^2]$ and $\mathbf{G}_2(\breve{x}_2) = [-2.6277\times 10^{-1}\breve{x}_2^2 + 5.6804\times 10^2 \quad 7.7914\times 10^{-2}\breve{x}_2^2 + 1.9794\times 10^2]$, and the polynomial observer gains are obtained as $\mathbf{L}_1(\breve{x}_2) = [8.3334\times 10^{-13}\breve{x}_2^2 + 1.5901\times 10 \quad 1.8529\times 10^{-11}\breve{x}_2^2 + 2.3319\times 10]^T$ and $\mathbf{L}_2(\breve{x}_2) = [6.6685\times 10^{-12}\breve{x}_2^2 + 1.5901\times 10 \quad 3.0865\times 10^{-11}\breve{x}_2^2 + 2.3319\times 10]^T$.

The above polynomial controller gains and polynomial observer gains are applied to the original dynamic system of the inverted pendulum (124). Considering 4 different initial conditions, the time response of system states are shown in Fig. 6 which shows that the inverted pendulum can be successfully stabilized. Choosing initiation conditions $\mathbf{x}(0) = [\frac{70\pi}{180} \quad 0]^T$ and $\breve{\mathbf{x}}(0) = [\frac{35\pi}{180} \quad 0]^T$ for demonstration, the estimated system states are shown in Fig. 7. The corresponding sampled output and control input are shown in Fig. 8. As is exhibited in Fig. 8(a), the measured output is kept to be constant during the sampling interval. Although the sampling activity increases the difficulty of controlling the inverted pendulum, the proposed polynomial fuzzy observer-controller can successfully stabilize the inverted pendulum using the sampled-output measurements.

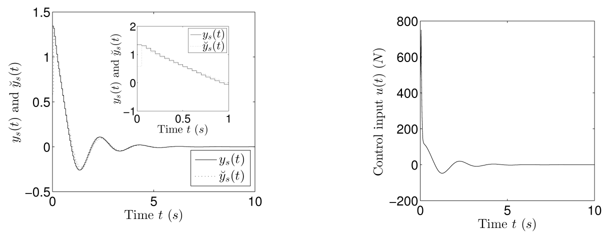
To compare the proposed control strategy with some relevant published papers, the polynomial fuzzy model used to represent the inverted pendulum is more general than T-S fuzzy model considered in [40]-[43], [50], [54]-[58]. The unmeasurable premise variables appeared in these examples provide more freedom for designing polynomial fuzzy model than measurable premise variables in [44]. Furthermore, one step design of the observer-controller is achieved instead of two steps [44] or iterative procedure [50]. The controller is allowed to be polynomial and the output matrix $\mathbf{C}$ is allowed to be different in each fuzzy rule, both of which are more general than [53]. Additionally, the maximum sampling interval 0.018 seconds achieved in [50] for the inverted pendulum is exceeded in this paper benefited from the continuous-time polynomial

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2015.2402685, IEEE Transactions on Fuzzy Systems



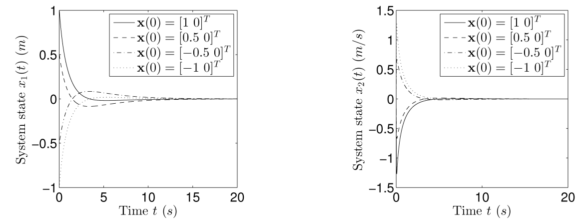
(a) Time response of $x_1(t)$ and $\breve{x}_1(t)$.
(b) Time response of $x_2(t)$ and $\breve{x}_2(t)$.

Fig. 7. Time response of system states and estimated states for $\mathbf{x}(0) = [\frac{70\pi}{180} \quad 0]^T$.



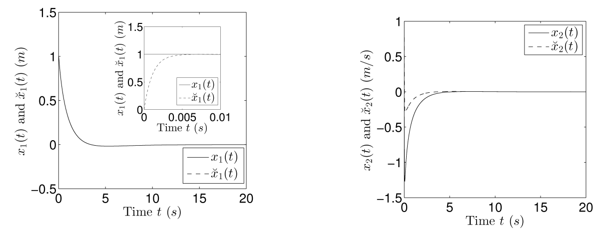
(a) Time response of $y_s(t)$ and $\breve{y}_s(t)$.
(b) Time response of $u(t)$.

Fig. 8. Time response of sampled output, estimated sampled output and control input for $\mathbf{x}(0) = [\frac{70\pi}{180} \quad 0]^T$.



(a) Time response of $x_1(t)$.
(b) Time response of $x_2(t)$.

Fig. 9. Time response of system states of the mass-spring-damper system with 4 different initial conditions.



(a) Time response of $x_1(t)$ and $\breve{x}_1(t)$.
(b) Time response of $x_2(t)$ and $\breve{x}_2(t)$.

Fig. 10. Time response of system states and estimated states for $\mathbf{x}(0) = [1 \quad 0]^T$.

fuzzy observer.

The computational time for checking the SOS conditions of Theorems 1 and 2 for the inverted pendulum are 57.236 seconds and 1994.563 seconds, receptively. The computational time for higher dimensional system may be more than the above values.

C. Nonlinear Mass-Spring-Damper System

We follow the same control strategy in previous examples to stabilize a nonlinear mass-spring-damper system whose dynamics is given by [61] and stated as follows:

$$M\ddot{x}(t) + g(x(t), \dot{x}(t)) + f(x(t)) = \phi(\dot{x}(t))u(t), \qquad (125)$$

where M is the mass; $g(x(t), \dot{x}(t)) = D(c_1 x(t) + c_2 \dot{x}(t)^3 + c_3(t)\dot{x}(t))$, $f(x(t)) = K(c_4 x(t) + c_5 x(t)^3 + c_6 x(t))$ and $\phi(\dot{x}(t)) = 1.4387 + c_7 \dot{x}(t)^2 + c_8 \cos(5\dot{x}(t))$ are the damper nonlinearity, the spring nonlinearity and the input nonlinearity, respectively; $M = D = K = 1, c_1 = 0, c_2 = 1, c_3 = -0.3, c_4 = 0.01, c_5 = 0.1, c_6 = 0.3, c_7 = -0.03, c_8 = 0.2$; and $u(t)$ is the force.

Time t is dropped from now for simplicity. Denoting x_1 and x_2 as x and $\dot{x}$, respectively, and $\mathbf{x} = [x_1 \quad x_2]^T$, we obtain the following state space form:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \frac{1}{M}(-g(x_1, x_2) - f(x_1) + \phi(x_2)u). \end{aligned} \qquad (126)$$

The nonlinear term $f_1(x_2) = \cos(5x_2)$ is represented by sector nonlinearity technique [6] as follows: $f_1(x_2) = \mu_{M_1^1}(x_2) f_{1_{min}} + \mu_{M_1^2}(x_2) f_{1_{max}}$, where $\mu_{M_1^1}(x_2) = \frac{f_1(x_2) - f_{1_{max}}}{f_{1_{min}} - f_{1_{max}}}$, $\mu_{M_1^2}(x_2) = 1 - \mu_{M_1^1}(x_2)$, $f_{1_{min}} = -1$, $f_{1_{max}} = 1$. As a result, the nonlinear mass-spring-damper system is described by a 2-rule polynomial fuzzy model. The system and input matrices in each rule are given by $\mathbf{A}_1(\mathbf{x}) = \mathbf{A}_2(\mathbf{x}) = \begin{bmatrix} 0 & 1 \\ a_1(x_1) & a_2(x_2) \end{bmatrix}$, $\mathbf{B}_1(x_2) = [0 \quad b_1(x_2)]^T$, and $\mathbf{B}_2(x_2) = [0 \quad b_2(x_2)]^T$, where $a_1(x_1) = -\frac{1}{M}(Dc_1 + K(c_4 + c_6) + Kc_5 x_1^2)$, $a_2(x_2) = -\frac{1}{M}(Dc_3 + Dc_2 x_2^2)$, $b_1(x_2) = \frac{1}{M}(1.4387 + c_7 x_2^2 + c_8 f_{1_{min}})$, $b_2(x_2) = \frac{1}{M}(1.4387 + c_7 x_2^2 + c_8 f_{1_{max}})$. In addition, the output matrices are $\mathbf{C}_1 = \mathbf{C}_2 = [1 \quad 0]$. The membership functions are $w_1(x_2) = \mu_{M_1^1}(x_2)$ and $w_2(x_2) = \mu_{M_1^2}(x_2)$.

It can be seen that the premise variable $f_1(x_2)$ depends on unmeasurable system state x_2, and thus Theorem 1 is employed to design a PFMB observer-controller to stabilize the nonlinear mass-spring-damper system. We choose $\mathbf{N}_k(\breve{x}_1)$ of degree 0 and 2, $\mathbf{M}_j(\breve{x}_1)$ of degree 0 and 2, and keep other settings the same as Section IV-A. The polynomial controller gains are obtained as $\mathbf{G}_1(\breve{x}_1) = [-1.4754 \times 10^{-1}\breve{x}_1^2 - 1.0447 \times 10 \quad -4.8074 \times 10^{-2}\breve{x}_1^2 - 3.3439 \times 10]$ and $\mathbf{G}_2(\breve{x}_1) = [-6.4731 \times 10^{-2}\breve{x}_1^2 - 9.8315 \times 10 \quad -3.9791 \times 10^{-2}\breve{x}_1^2 - 3.3442 \times 10]$, and the polynomial observer gains are obtained as $\mathbf{L}_1(\breve{x}_2) = [4.1689 \times 10^{-3}\breve{x}_1^2 + 9.2052 \times 10^2 \quad 4.1692 \times 10^{-3}\breve{x}_1^2 + 1.0648 \times 10^3]^T$ and $\mathbf{L}_2(\breve{x}_2) = [4.1689 \times 10^{-3}\breve{x}_1^2 + 9.2052 \times 10^2 \quad 4.1692 \times 10^{-3}\breve{x}_1^2 + 1.0648 \times 10^3]^T$.

Considering 4 different initial conditions, the time response of system states are shown in Fig. 9 which shows that the nonlinear mass-spring-damper system can be stabilized by the designed polynomial fuzzy observer-controller. Choosing initiation conditions $\mathbf{x}(0) = [1 \quad 0]^T$ and $\breve{\mathbf{x}}(0) = [0 \quad 0]^T$ as an example, the estimated system states are shown in Fig. 10. Consequently, it is feasible to apply the proposed PFMB observer-control strategy for stabilization of nonlinear systems.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2015.2402685, IEEE Transactions on Fuzzy Systems

The MATLAB codes for these simulation examples can be downloaded by the following link: http://www.inf.kcl.ac.uk/staff/hklam/docs/MatlabCodes(paper105).zip. Readers may use the codes to easily implement the proposed polynomial fuzzy observer-controllers.

V. Conclusion

In this paper, the stability of PFMB observer-control system has been investigated. Two classes of PFMB observer-controllers have been considered. The first class considers continuous system output in the design while the second class considers the sampled-output measurements. In both classes, the polynomial controller gains and polynomial observer gains are allowed to be a function of estimated states. Moreover, the premise variables are allowed to be unmeasurable which complicates the stability analysis but enhances the applicability of the proposed PFMB observer-control scheme. Matrix decoupling technique has been employed in the stability analysis to obtain convex SOS stability conditions. Simulation examples have been presented to verify the stability analysis results and demonstrate the effectiveness of the proposed PFMB observer-control scheme.

References

[1] T. Takagi and M. Sugeno, "Fuzzy identification of systems and its applications to modelling and control," *IEEE Trans. Syst., Man, Cybern.*, vol. SMC-15, no. 1, pp. 116–132, Jan. 1985.

[2] M. Sugeno and G. T. Kang, "Structure identification of fuzzy model," *Fuzzy Sets Syst.*, vol. 28, no. 1, pp. 15–33, Oct. 1988.

[3] K. Tanaka, H. Yoshida, H. Ohtake, and H. O. Wang, "A sum of squares approach to modeling and control of nonlinear dynamical systems with polynomial fuzzy systems," *IEEE Trans. Fuzzy Syst.*, vol. 17, no. 4, pp. 911–922, Aug. 2009.

[4] K. Tanaka, H. Ohtake, and H. O. Wang, "Guaranteed cost control of polynomial fuzzy systems via a sum of squares approach," *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 39, no. 2, pp. 561–567, Apr. 2009.

[5] K. Tanaka and H. O. Wang, *Fuzzy Control Systems Design and Analysis: a Linear Matrix Inequality Approach*. New York: Wiley-Interscience, 2001.

[6] A. Sala and C. Ariño, "Polynomial fuzzy models for nonlinear control: a Taylor-series approach," *IEEE Trans. Fuzzy Syst.*, vol. 17, no. 6, pp. 284–295, Dec. 2009.

[7] H. O. Wang, K. Tanaka, and M. F. Griffin, "An approach to fuzzy control of nonlinear systems: stability and design issues," *IEEE Trans. Fuzzy Syst.*, vol. 4, no. 1, pp. 14–23, Feb. 1996.

[8] K. Tanaka, T. Ikeda, and H. O. Wang, "Fuzzy regulators and fuzzy observers: relaxed stability conditions and LMI-based designs," *IEEE Trans. Fuzzy Syst.*, vol. 6, no. 2, pp. 250–265, May 1998.

[9] S. Prajna, A. Papachristodoulou, and P. A. Parrilo, "Nonlinear control synthesis by sum-of-squares optimization: a Lyapunov-based approach," in *Proc. Asian Control Conf. (ASCC)*, vol. 1, Melbourne, Australia, Feb. 2004, pp. 157–165.

[10] G. Feng, "A survey on analysis and design of model-based fuzzy control systems," *IEEE Trans. Fuzzy Syst.*, vol. 14, no. 5, pp. 676–697, Oct. 2006.

[11] A. Sala, "On the conservativeness of fuzzy and fuzzy-polynomial control of nonlinear systems," *Annu. Rev. Control*, vol. 33, no. 1, pp. 48–58, 2009.

[12] X. Liu and Q. Zhang, "Approaches to quadratic stability conditions and H_∞ control designs for Takagi-Sugeno fuzzy systems," *IEEE Trans. Fuzzy Syst.*, vol. 11, no. 6, pp. 830–839, Dec. 2003.

[13] C. H. Fang, Y. S. Liu, S. W. Kau, L. Hong, and C. H. Lee, "A new LMI-based approach to relaxed quadratic stabilization of Takagi-Sugeno fuzzy control systems," *IEEE Trans. Fuzzy Syst.*, vol. 14, no. 3, pp. 386–397, Jun. 2006.

[14] A. Sala and C. Ariño, "Asymptotically necessary and sufficient conditions for stability and performance in fuzzy control: applications of Polya's theorem," *Fuzzy Sets Syst.*, vol. 158, no. 24, pp. 2671–2686, Jul. 2007.

[15] J. C. Lo and J. R. Wan, "Studies on linear matrix inequality relaxations for fuzzy control systems via homogeneous polynomials," *IET Control Theory Applicat.*, vol. 4, no. 11, pp. 2293–2302, Nov. 2010.

[16] H. Ohtake, K. Tanaka, and H. O. Wang, "Switching fuzzy controller design based on switching Lyapunov function for a class of nonlinear systems," *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 36, no. 1, pp. 13–23, Feb. 2006.

[17] Y. J. Chen, H. Ohtake, K. Tanaka, W. J. Wang, and H. O. Wang, "Relaxed stabilization criterion for T-S fuzzy systems by minimum-type piecewise Lyapunov function based switching fuzzy controller," *IEEE Trans. Fuzzy Syst.*, vol. 120, no. 6, pp. 1166–1173, Dec. 2012.

[18] H. K. Lam, M. Narimani, H. Li, and H. Liu, "Stability analysis of polynomial-fuzzy-model-based control systems using switching polynomial Lyapunov function," *IEEE Trans. Fuzzy Syst.*, vol. 21, no. 5, pp. 800–813, Oct. 2013.

[19] M. Bernal and T. Guerra, "Generalized nonquadratic stability of continuous-time Takagi-Sugeno models," *IEEE Trans. Fuzzy Syst.*, vol. 18, no. 4, pp. 815–822, Aug. 2010.

[20] M. Bernal, A. Sala, A. Jaadari, and T.-M. Guerra, "Stability analysis of polynomial fuzzy models via polynomial fuzzy Lyapunov functions," *Fuzzy Sets Syst.*, vol. 185, no. 1, pp. 5–14, Dec. 2011.

[21] H. K. Lam and J. Lauber, "Membership-function-dependent stability analysis of fuzzy-model-based control systems using fuzzy Lyapunov functions," *Inform. Sci.*, vol. 232, pp. 253–266, May 2013.

[22] G. Feng, "H_∞ controller design of fuzzy dynamic systems based on piecewise Lyapunov functions," *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 34, no. 1, pp. 283–292, Feb. 2004.

[23] G. Feng, C. L. Chen, D. Sun, and Y. Zhu, "H_∞ controller synthesis of fuzzy dynamic systems based on piecewise Lyapunov functions and bilinear matrix inequalities," *IEEE Trans. Fuzzy Syst.*, vol. 13, no. 1, pp. 94–103, Feb. 2005.

[24] K. Guelton, N. Manamanni, C. C. Duong, and D. L. Koumba Emianiwe, "Sum-of-squares stability analysis of Takagi-Sugeno systems based on multiple polynomial Lyapunov functions," *Int. J. Fuzzy Syst.*, vol. 15, no. 1, pp. 34–41, Mar. 2013.

[25] H. K. Lam and L. D. Seneviratne, "Stability analysis of polynomial fuzzy-model-based control systems under perfect/imperfect premise matching," *IET Control Theory Applicat.*, vol. 5, no. 15, pp. 1689–1697, Oct. 2011.

[26] H. K. Lam and S.-H. Tsai, "Stability analysis of polynomial-fuzzy-model-based control systems with mismatched premise membership functions," *IEEE Trans. Fuzzy Syst.*, vol. 22, no. 1, pp. 223–229, Feb. 2014.

[27] A. Sala and C. Ariño, "Relaxed stability and performance LMI conditions for Takagi-Sugeno fuzzy systems with polynomial constraints on membership function shapes," *IEEE Trans. Fuzzy Syst.*, vol. 16, no. 5, pp. 1328–1336, Oct. 2008.

[28] M. Narimani and H. K. Lam, "SOS-based stability analysis of polynomial fuzzy-model-based control systems via polynomial membership functions," *IEEE Trans. Fuzzy Syst.*, vol. 18, no. 5, pp. 862–871, Oct. 2010.

[29] H. K. Lam, "Polynomial fuzzy-model-based control systems: stability analysis via piecewise-linear membership functions," *IEEE Trans. Fuzzy Syst.*, vol. 19, no. 3, pp. 588–593, Jun. 2011.

[30] M. Bernal, T. M. Guerra, and A. Kruszewski, "A membership-function-dependent approach for stability analysis and controller synthesis of Takagi-Sugeno models," *Fuzzy Sets Syst.*, vol. 160, no. 19, pp. 2776–2795, 2009.

[31] H. K. Lam and M. Narimani, "Quadratic stability analysis of fuzzy-model-based control systems using staircase membership functions," *IEEE Trans. Fuzzy Syst.*, vol. 18, no. 1, pp. 125–137, Feb. 2010.

[32] H. K. Lam, "LMI-based stability analysis for fuzzy-model-based control systems using artificial T-S fuzzy model," *IEEE Trans. Fuzzy Syst.*, vol. 19, no. 3, pp. 505–513, Jun. 2011.

[33] S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*. Society for Industrial and Applied Mathematics (SIAM), 1994.

[34] H. K. Lam, F. H. F. Leung, and P. K. S. Tam, "Fuzzy control of a class of multivariable nonlinear systems subject to parameter uncertainties: model reference approach," *Int. J. Approximate Reasoning*, vol. 26, no. 2, pp. 129–144, 2001.

[35] H. K. Lam and W. K. Ling, "Sampled-data fuzzy controller for continuous nonlinear systems," *IET Control Theory Applicat.*, vol. 2, no. 1, pp. 32–39, Jan. 2008.

[36] H. K. Lam and L. D. Seneviratne, "Tracking control of sampled-data fuzzy-model-based control systems," *IET Control Theory Applicat.*, vol. 3, no. 1, pp. 56–67, Jan. 2009.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2015.2402685, IEEE Transactions on Fuzzy Systems

[37] H. K. Lam and J. C. Lo, "Output regulation of polynomial-fuzzy-model-based control systems," *IEEE Trans. Fuzzy Syst.*, vol. 2, no. 21, pp. 262–274, Apr. 2013.

[38] J. Yoneyama, M. Nishikawa, H. Katayama, and A. Ichikawa, "Design of output feedback controllers for Takagi-Sugeno fuzzy systems," *Fuzzy Sets Syst.*, vol. 121, no. 1, pp. 127–148, 2001.

[39] S. K. Nguang and P. Shi, "H_∞ fuzzy output feedback control design for nonlinear systems: an LMI approach," *IEEE Trans. Fuzzy Syst.*, vol. 11, no. 3, pp. 331–340, Jun. 2003.

[40] C. S. Tseng and B. S. Chen, "Robust fuzzy observer-based fuzzy control design for nonlinear discrete-time systems with persistent bounded disturbances," *IEEE Trans. Fuzzy Syst.*, vol. 17, no. 3, pp. 711–723, Jun. 2009.

[41] T. M. Guerra, A. Kruszewski, L. Vermeiren, and H. Tirmant, "Conditions of output stabilization for nonlinear models in the Takagi-Sugeno's form," *Fuzzy Sets Syst.*, vol. 157, no. 9, pp. 1248–1259, 2006.

[42] M. H. Asemani and V. J. Majd, "A robust observer-based controller design for uncertain T-S fuzzy systems with unknown premise variables via LMI," *Fuzzy Sets Syst.*, vol. 212, no. 0, pp. 21–40, 2013.

[43] X. H. Chang and G. H. Yang, "A descriptor representation approach to observer-based control synthesis for discrete-time fuzzy systems," *Fuzzy Sets Syst.*, vol. 185, no. 1, pp. 38–51, 2011.

[44] K. Tanaka, H. Ohtake, T. Seo, M. Tanaka, and H. O. Wang, "Polynomial fuzzy observer designs: a sum-of-squares approach," *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 42, no. 5, pp. 1330–1342, Oct. 2012.

[45] S. Lall and G. Dullerud, "An LMI solution to the robust synthesis problem for multi-rate sampled-data systems," *Automatica*, vol. 37, no. 12, pp. 1909–1922, 2001.

[46] L. S. Hu, J. Lam, Y. Y. Cao, and H. H. Shao, "A linear matrix inequality (LMI) approach to robust H_2 sampled-data control for linear uncertain systems," *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 33, no. 1, pp. 149–155, Feb. 2003.

[47] E. Fridman, A. Seuret, and J. P. Richard, "Robust sampled-data stabilization of linear systems: an input delay approach," *Automatica*, vol. 40, no. 8, pp. 1441–1446, 2004.

[48] D. W. Kim and H. J. Lee, "Sampled-data observer-based output-feedback fuzzy stabilization of nonlinear systems: exact discrete-time design approach," *Fuzzy Sets Syst.*, vol. 201, no. 0, pp. 20–39, 2012.

[49] H. K. Lam and F. H. F. Leung, "Sampled-data fuzzy controller for time-delay nonlinear system: LMI-based and fuzzy-model-based approaches," *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 37, no. 3, pp. 617–629, Jun. 2007.

[50] H. Gao and T. Chen, "Stabilization of nonlinear systems under variable sampling: a fuzzy control approach," *IEEE Trans. Fuzzy Syst.*, vol. 15, no. 5, pp. 972–983, Oct. 2007.

[51] H. K. Lam, "Sampled-data fuzzy-model-based control systems: stability analysis with consideration of analogue-to-digital converter and digital-to-analogue converter," *IET Control Theory Applicat.*, vol. 4, no. 7, pp. 1131–1144, Jul. 2010.

[52] X. L. Zhu, B. Chen, D. Yue, and Y. Wang, "An improved input delay approach to stabilization of fuzzy systems under variable sampling," *IEEE Trans. Fuzzy Syst.*, vol. 20, no. 2, pp. 330–341, Apr. 2012.

[53] H. K. Lam, "Stabilization of nonlinear systems using sampled-data output-feedback fuzzy controller based on polynomial-fuzzy-model-based control approach," *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 42, no. 1, pp. 258–267, Feb. 2012.

[54] S. K. Nguang and P. Shi, "Fuzzy H_∞ output feedback control of nonlinear systems under sampled measurements," *Automatica*, vol. 39, no. 12, pp. 2169–2174, 2003.

[55] H. Zhang, H. Yan, Q. Chen, and T. Liu, "Quantised H_∞ control for sampled fuzzy systems," *IET Control Theory Applicat.*, vol. 6, no. 17, pp. 2686–2695, Nov. 2012.

[56] C. P. G. Flores, B. C. Toledo, J. P. G. Sandoval, S. D. Gennaro, and V. G. lvarez, "A reset observer with discrete/continuous measurements for a class of fuzzy nonlinear systems," *J. Franklin Inst.*, vol. 350, no. 8, pp. 1974–1991, 2013.

[57] H. Li, X. Sun, H. R. Karimi, and B. Niu, "Dynamic output-feedback passivity control for fuzzy systems under variable sampling," *Math. Problems Eng.*, vol. 2013, 2013.

[58] H. Li, X. Jing, H. K. Lam, and P. Shi, "Fuzzy sampled-data control for uncertain vehicle suspension systems," *IEEE Trans. Cybern.*, vol. PP, no. 99, pp. 1–1, Sept. 2013.

[59] L. Xie and C. E. De Souza, "Robust H_∞ control for linear systems with norm-bounded time-varying uncertainty," *IEEE Trans. Autom. Control*, vol. 37, no. 8, pp. 1188–1191, Aug. 1992.

[60] K. Gu, "An integral inequality in the stability problem of time-delay systems," in *2000 Proc. 39th IEEE Conf. Decision and Control*, vol. 3, 2000, pp. 2805–2810.

[61] H. K. Lam, F. H. F. Leung, and P. K. S. Tam, "Stable and robust fuzzy control for uncertain nonlinear systems," *IEEE Trans. Syst., Man, Cybern. A, Syst., Humans*, vol. 30, no. 6, pp. 825–840, Nov. 2000.

Chuang Liu received the BEng degree in Mechanical Engineering from Tsinghua University, Beijing, China, in 2011, and the MSc degree in Robotics from King's College London, London, U.K., in 2013.

He is currently a PhD student at King's College London. His research interests include fuzzy-model-based control and its applications.

H.K. Lam (M'98-SM'10) received the B.Eng. (Hons.) and Ph.D. degrees from the Department of Electronic and Information Engineering, The Hong Kong Polytechnic University, Hong Kong, in 1995 and 2000, respectively.

From 2000 to 2005, he was a Postdoctoral Fellow and a Research Fellow with the Department of Electronic and Information Engineering, The Hong Kong Polytechnic University, respectively. In 2005, he joined Kings College London, London, U.K., as a Lecturer and currently is Reader.

His current research interests include intelligent control systems and computational intelligence. He has served as a program committee member and international advisory board member for various international conferences and a reviewer for various books, international journals and international conferences. He is an associate editor for IEEE Transactions on Fuzzy Systems, IET Control Theory & Applications, International Journal of Fuzzy Systems and Neurocomputing; and a guest editor for a number of international journals. He is in the editorial boards of a number of international journals.

He is the coeditor for two edited volumes: Control of Chaotic Nonlinear Circuits (World Scientific, 2009) and Computational Intelligence and Its Applications (World Scientific, 2012), and the coauthor of the book Stability Analysis of Fuzzy-Model-Based Control Systems (Springer, 2011).